Time Variation in Expected Returns and Aggregate Asset Growth

Min Kim

Department of Finance and Business Economics, Marshall School of Business
University of Southern California

January 20, 2009

Abstract

Aggregate asset growth—first differences of the logarithm of household net worth—can capture time variation in changes in expected returns in quarterly and annual horizons in which stock returns have virtually zero autocorrelations. Regressions of changes in stock returns on aggregate asset growth provide stable estimates of slope coefficients over time, which improve out-of-sample predictability. In particular, aggregate asset growth performs better out of sample in terms of a mean squared predictive error suggested by Clark and West (2007) than other predictors, such as the dividend-to-price ratio, “cay” and labor income growth.

Key words: conditional expected return, predictive regression, aggregate asset growth, out-of-sample forecast

0 I am very grateful to Wayne Ferson and Ravi Jagannathan for their illuminating suggestions and comments. All errors are my own.

Email address: minskim@usc.edu (Min Kim).
Introduction

This paper proposes a new approach for the study of time variation in expected stock returns using macroeconomic variables. Looking at aggregate asset growth, specifically, first differences of the logarithm of household net worth, provides powerful and simple univariate estimates of changes in expected returns on stocks in quarterly and annual horizons, in which stock returns have virtually zero autocorrelations. OLS regression of changes in stock returns on asset growth leads to stable estimates of slope coefficients over time, which improve out-of-sample predictability. Moreover, in terms of a mean squared predictive error suggested by Clark and West (2007), only asset growth predicts significantly better than the historical mean model over an out-of-sample period except when the markets were extremely volatile.¹

To examine time variation in expected stock returns, we note that the first differences of expected returns—changes in expected returns—can be estimated using the first differences of macroeconomic variables. This does not imply that the levels of expected returns can be estimated by the levels of macroeconomic variables. An extension of the model by Campbell and Mankiw (1989) suggests that changes in the consumption-wealth ratio are related to changes in expected returns on the market portfolio (aggregate assets). The approximate components of the changes in the consumption-wealth ratio—the growth rates of consumption, asset holdings and labor income—convey relevant information about changes in expected returns on the market portfolio.

However, as long as consumption growth and labor income growth are not too volatile, the main component driving changes in the consumption-wealth ratio is the growth of asset holdings. Changes in expected returns on the market portfolio, then, can be estimated by using the growth of aggregate asset holdings. When agents expect future returns to be higher, the positive shocks to discount rates decrease current value of asset holdings.² In essence, a negative asset growth today is associated with an increase in expected future returns.

In addition to the model’s implications for changes in stock returns, looking at the changes is also useful from the econometric point of view when expected returns are highly persistent but realized returns have few autocorrelations.³

¹ Two recent studies show that diversifying across individual forecasts (Rapach, David, Jack Strauss and Guofu Zhou (2008)) and forecasting separately dividend yield, earnings growth, and price-earnings ratio growth (Ferreira and Santa-Clara (2008)) can outperform the historical mean model.
³ Many researchers model expected returns as a highly persistent process. Yet, their
If stock returns were serially correlated, we could use the levels of returns to capture time variation in expected returns. Otherwise, given that expected returns are highly persistent, taking first differences of stock returns is an alternative. In this case, changes in returns in the next period are roughly the sum of three terms (ignoring the term involved with expected returns due to their persistency): Shocks to expected returns, unexpected returns in the next period, and minus unexpected returns today. A variable that is correlated with shocks to expected returns, thus, can be used to predict changes in returns. Asset holdings today, for example, decrease with shocks to expected returns but increase with unexpected returns today. As a result, asset growth is negatively related with shocks to expected returns while it is positively related with shocks to returns. Therefore, we can use asset growth to predict changes in returns and thereby future returns.

In-sample empirical results support the predictive ability of asset growth as discussed above. I proxy the market portfolio using CRSP value weighted index (CRSP VW). Although CRSP VW does not include all assets in the market portfolio but only public stocks, the index must be highly correlated with the market portfolio and may represent it. Many studies also use the index to proxy the unobservable market portfolio. Thus, based on the implications of the model, I regress changes in returns on CRSP VW one period ahead against asset growth this period. The regression coefficients of changes in stock returns one period ahead on asset growth are statistically significant for both quarterly and annual horizons. Their magnitudes are economically significant. For example, a 1% increase in aggregate asset holdings (1% asset growth) predicts that returns on CRSP VW will decrease by around 3.6% in the next quarter.

The estimated changes in expected returns by asset growth explain around 39% and 46% of variations in quarterly and annual changes in returns respectively. Adding other variables, such as consumption growth and labor income growth, does not improve the predictive power of asset growth. This confirms that consumption growth and labor income growth do not have much information relevant to changes in expected returns.

Moreover, the coefficient estimates of asset growth are not sensitive to sample periods. The stable magnitudes are consistent with the model’s implication that the slopes should be equal to the reciprocal of the steady-state investment ratio. The estimated investment ratio is 0.28. The stable estimates improve out-of-sample predictability. Tests show that using aggregate asset growth autocorrelations over quarterly and annual horizons are very small. For instance, they are around 0.03 with the standard error 0.07 from the fourth quarter of 1952 to the fourth quarter of 2006. Despite the small estimates of autocorrelations over the sample period, the implied persistency of expected returns is 0.98, given other sample estimates. See Section 2.2 for details.
provides unbiased predictions for out-of-sample changes in stock returns.

I also compare out-of-sample predictive ability of various variables—such as the dividend-to-price ratio, “cay” and labor income growth—in terms of a mean squared predictive error suggested by Clark and West (2007). The comparison indicates that over the period from the first quarter of 1980 to the fourth quarter of 2006, the variable “cay”—estimated using the whole sample period—performs better than the null model that expected returns are constant. However, excluding the period when the markets were extremely volatile (from the third quarter of 2000 and the third quarter of 2002), aggregate asset growth beats the null model while “cay” becomes insignificant. Moreover, in recent years (after the third quarter of 2002) only asset growth predicts better than the null model.

These empirical findings suggest a few advantages to using aggregate asset growth to examine time variation in the expected returns on stocks. First, asset growth can be used to predict stock returns for intermediate horizons such as quarterly and annual horizons, in which stock returns exhibit little time dependency. Asset growth—available with quarterly and annual frequencies—can be used to predict changes in returns and then we can predict subsequent stock returns by adding predicted changes to current stock returns. Second, the regression coefficient of aggregate asset growth is stable across sample periods and returns horizons. This should enhance predictive ability out of sample.

The rest of this paper is organized as follows: Section 1 provides the literature review; Section 2 derives the relationship between aggregate asset growth and expected returns on assets; a discussion of the empirical tests follows these sections; Section 4 conducts out-of-sample tests; Section 5 discusses robustness checks; and a comparison of predictive ability follows in Section 6. Finally, we conclude and provide an appendix with technical details.

4 “cay” is the cointegrating relationship of consumption, asset holdings and labor income as constructed by Lettau and Ludvigson (2001).
5 Santos and Veronesi (2006) show predictive ability of the labor income to consumption ratio for stock returns.
6 Avramov (2002) and Goyal and Welch (2007) show that if we estimate the cointegrating relationship using a recursive method, the variable cay is not as good as in the case of estimating the relationship using the whole sample period. See also Brennan and Xia (2005).
1 Literature Review

Although this paper estimates changes in the expected returns, it is based on a view that many studies in the predictive regression literature adopt: Time variation in expected returns in efficient markets can result in predictable components of stock returns.\textsuperscript{7} Fama and French (1988) provide straightforward, intuitive support for this view. Suppose investors react rationally to time varying investment opportunity sets and their expectation of returns is persistent but stationary. Then shocks to expected returns will have an effect on expected returns in the near future, but not in the distant future. If these shocks are not correlated with shocks to dividends, they lead to opposite shocks to current prices and accordingly to current returns—thereby creating a mechanism by which stock returns may be predicted.

There are three main approaches in investigating time variation in expected returns and its implications for variation in realized returns. One approach is to test for the presence of the stationary component in stock prices. This approach typically assumes a specific stationary process for the expected returns and uses autocorrelations in stock returns.\textsuperscript{8} Alternatively, many studies use financial variables such as dividend-price ratios to estimate expected returns. The third approach is based on the fact that stock returns appear to vary with business cycles and uses macroeconomic variables associated with business cycles.

Fama and French and Poterba and Summers (1998) are among the first researchers to examine a stationary component in stock prices. Their findings show that the time-varying expected returns due to the stationary component of prices may account for a large fraction of return variation for more than a one-year holding period. In particular, Fama and French argue that if prices have both a random walk and a stationary component following an AR (1) process, we may observe a U-shaped pattern of the regression coefficients of current returns on past returns. In predicting returns over short horizons, the covariance between changes in the stationary component of prices is close to zero, but in considering longer horizons, the covariance becomes negative. However, as the horizons get longer, the variance of changes in the random walk component dominates the variance of changes in the stationary component. This results in zero slopes of past returns in the regressions for current returns at very long horizons.

\textsuperscript{7} Fama (1976) provides a graphical illustration and Ferson and Gibbons (1985) note that conditional expected returns that are varying over time are not inconsistent with the hypothesis that the unconditional mean is constant.\textsuperscript{8} Poterba and Summers (1998) provide an excellent summary of test methods for the presence of stationary components of prices.
Alternatively, other studies use financial variables such as dividend-price ratios and earnings-price ratios to capture time variation in expected returns. Although those variables show economically significant predictive power for stock returns, the statistical significance of financial variables is often marginal. Their forecasting ability is particularly weak for returns over short and intermediate horizons. Nevertheless, Cochrane (2006) provides indirect evidence for predictive ability of dividend-price ratios. If the dividend-price ratio does not predict stock returns or dividend growth, we should observe a constant dividend-price ratio. Since dividend growth is not predictable by the dividend-price ratio, the variation in the dividend-price ratio, therefore, suggests that it explains the variation in stock returns.

A model for the implication of macroeconomic variables for expected stock returns is suggested by Campbell and Mankiw (1989). These researchers derive the relationship between the consumption-wealth ratio and expected returns on wealth, based only on the budget constraint of an optimal consumption problem. Despite their theoretical appeal that the results hold for a general class of investors’ preferences, macroeconomic variables have not been very successfully linked with stock returns. This is because most of the relevant variables—including consumption and labor income—are not stationary and their growth rates are empirically shown to have little predictive power for stock returns. On the other hand, Lettau and Ludvigson (2001) show that consumption, asset holdings and labor income share common trends, and that the variable “cay” can predict stock returns over intermediate horizons.

Although all three approaches provide evidence that expected returns vary through time, few are very successful at predicting stock returns over intermediate horizons. Only in long horizons and for the sample period including pre-1940 do the tests of Fama and French show significant support for a stationary component in prices. This may be because a stationary but highly persistent component in prices results in autocorrelations of stock returns that are close to zero in intermediate-horizons, as Fama and French suggest. They also argue that after 1940, stock returns appear close to white noise. Ferson, Heuson and Su (2005) find no evidence of “weak-form” predictability of monthly returns on individual stocks in recent sample periods.

Similarly, financial variables are most successful at predicting returns over long horizons. mention Goyal and Welch and Ferson, Sarkissian and Simin.

Finally, although Lettau and Ludvigson show evidence of predictive power of the variable “cay” among the macroeconomic variables for intermediate-horizon returns, the predictive power is sensitive to sample periods. More-

---


10 Lettau and Ludvigson show that the cointegrating relationship, the variable
over, as I discuss in the next section, their theoretical model is based on a poor approximation in deriving the relationship between the consumption-wealth ratio and the returns on assets.

2 Expected returns and consumption-wealth ratio

Assume that aggregate wealth at time $t$ ($W_t$) comprises asset holdings ($A_t$) and human capital ($H_t$) and that both are tradable. Let us denote a variable in logarithm by a lower case letter. For example, $w_t$ is the logarithm of $W_t$, $r_{w,t+i}$ is the logarithm of the gross return on wealth for one period from $t+i-1$ to $t+i$, and $c_t$ is the logarithm of consumption $C_t$ in the period $t$. As shown in the appendix, $c_t - w_t$, the logarithm of the consumption-wealth ratio ($\frac{C_t}{W_t}$), may be approximated as

$$c_t - w_t \approx E_t \left[ \sum_{i=1}^{\infty} \rho_w^i (r_{w,t+i} - \Delta c_{t+i}) \right]$$  

(1)

where $E_t$ denotes the expectation conditional on information in the period $t$ and $\rho_w$ is the steady-state level of investment to aggregate wealth ($\frac{W-C}{W}$).

Similarly, changes in the consumption-wealth ratio can be approximated as

$$\Delta c_t - \Delta w_t \approx E_t \left[ \sum_{i=1}^{\infty} \rho_w^i (r_{w,t+i} - \Delta c_{t+i}) \right] - E_{t-1} \left[ \sum_{i=1}^{\infty} \rho_w^i (r_{w,t-1+i} - \Delta c_{t-1+i}) \right].$$  

(2)

As the appendix shows, given that asset holdings to wealth ratio ($\frac{A_t}{W_t}$) does not vary much from the steady-state level ($\lambda \equiv \frac{Y}{W}$) and that the human capital to labor income ratio ($\frac{H_t}{Y_t}$) is a stationary process, denoted by $Z_t$, we can express changes in wealth as

$$\Delta w_t \approx \lambda \Delta a_t + (1 - \lambda)(\Delta y_t + \Delta z_t)$$  

(3)

where $y_t$ and $z_t$ are the logarithms of human capital $Y_t$ and the stationary process $Z_t$ respectively.

"cay", can achieve an adjusted R-squared as high as 10% for the period of the fourth quarter of 1952 to the third quarter of 1998. However, the adjusted R-squared, for example, drops to 7% for the period of the first quarter of 1951 to the fourth quarter of 2003 and to 4% for the period of the first quarter of 1978 to the fourth quarter of 2003 (Pastor and Stambaugh (2006)).

11 If we view labor income as dividends of human capital as Campbell (1996) and Jagannathan and Wang (1996) do, the human capital to labor income ratio can be compared to price to dividend ratio of stocks.
Furthermore, Campbell (1996) shows the approximation of the return on wealth, 
\[ r_{w,t+i} \approx \lambda r_{a,t+i} + (1 - \lambda) r_{h,t+i}, \]
where \( r_{a,t} \) is the return on assets and \( r_{h,t} \) the return on human capital. This, along with (3), allows us to rewrite the equation (2) as

\[
\Delta c_t - (\lambda \Delta a_t + (1 - \lambda) \Delta y_t) \\
\approx E_t \left[ \sum_{i=1}^{\infty} \rho^i_w (\lambda r_{a,t+i} + (1 - \lambda) r_{h,t+i} - \Delta c_{t+i}) \right] - \\
E_{t-1} \left[ \sum_{i=1}^{\infty} \rho^i_w (\lambda r_{a,t-1+i} + (1 - \lambda) r_{h,t-1+i} - \Delta c_{t-1+i}) \right] + (1 - \lambda) \Delta z_t. \quad (4)
\]

For comparison, the equation (5) shows the model in Lettau and Ludvigson (2001),

\[
c_t - (\lambda a_t + (1 - \lambda) y_t) \approx E_t \left[ \sum_{i=1}^{\infty} \rho^i_w (\lambda r_{a,t+i} + (1 - \lambda) r_{h,t+i} - \Delta c_{t+i}) \right] + (1 - \lambda) \Delta z_t,
\]

which suggests that the deviation from the cointegrating relationship among consumption, asset holdings and labor income can predict future asset returns provided that expected future returns on human capital (\( r_{h,t} \)) and consumption growth (\( \Delta c_t \)) do not vary much. However, this predictive relationship is derived by approximating wealth as a convex combination of its two components, asset holdings and human capital. While the return on wealth is a convex combination, the level of wealth is the sum of the two components and its level must exceed that of each component. The corrected model is suggested by the equation (4).

As employed by Lettau and Ludvigson, suppose the return on human capital (\( r_{h,t} \)) and consumption growth (\( \Delta c_t \)) do not vary much from period to period.\(^{12}\) Then, conditional expectations of them vary even less and thus we may assume \( E_t[r_{h,t+i} - E_{t-1}[r_{h,t-1+i}] \approx 0 \) and \( E_t[\Delta c_{t+i}] - E_{t-1}[\Delta c_{t-1+i}] \approx 0.^{13} \) Moreover, if \( r_{h,t} \) does not vary much, labor income growth (\( \Delta y_t \)) does not change much either since labor income is dividends on human capital as we assumed earlier. Thus, we also have \( E_t[\Delta y_{t+i}] - E_{t-1}[\Delta y_{t-1+i}] \approx 0 \), which leads

\(^{12}\) Over the fourth quarter of 1951 to the fourth quarter of 2005, consumption growth has mean 0.0051 and standard deviation 0.0045. Labor income growth (proxy for return on human capital) has the sample mean 0.0060 and the standard deviation 0.0089. They are much less volatile compared to asset growth, of which mean is 0.06 and the standard deviation 0.02.

\(^{13}\) Variation in expected consumption growth may not be significant over a short-term period, such as quarterly and annula horizonsis. This is not inconsistent with the long-run risk literature, which is originated by Bansal and Yaron (2003).
to $\Delta z_t = 0$. As a result, we can further simplify the expression (4) to arrive at

$$
\Delta c_t - (\lambda \Delta a_t + (1 - \lambda) \Delta y_t) \approx \sum_{i=1}^{\infty} \rho_i \lambda \{E_t[r_{a,t+i}] - E_{t-1}[r_{a,t-1+i}]) \}. \quad (6)
$$

3 Predictive regressions

3.1 Methodology

Let us write returns on stocks as

$$
r_{t+1} = \mu_t + \varepsilon_{t+1} \quad (7)
$$

where $\mu_t \equiv E_t[r_{t+1}]$ and unexpected returns, $\varepsilon_{t+1}$, are white noise with $E_t[\varepsilon_{t+1}|\mu_t] = 0$. Assume that the expected returns $\mu_t$ follow an autoregressive process of the order one as

$$
\mu_t = \phi \mu_{t-1} + v_t \quad (8)
$$

where the persistency parameter satisfies the stationary condition, $|\phi| < 1$. Shocks to expected returns, $v_t$, are white noise and $E_{t-1}[v_t|\mu_{t-1}] = 0$.

On the other hand, changes in returns are

$$
r_{t+1} - r_t = -(1-\phi)\mu_{t-1} + v_t - \varepsilon_t + \varepsilon_{t+1}. \quad (9)
$$

In particular, when expected returns are highly persistent ($\phi \approx 1$), we may have

$$
r_{t+1} - r_t \approx v_t - \varepsilon_t + \varepsilon_{t+1}. \quad (10)
$$

Note that by our assumptions, $v_t - \varepsilon_t$ are serially uncorrelated.

As the equation (6) suggests, we can capture changes in expected returns, $-(1-\phi)\mu_{t-1} + v_t$ or $v_t$, by changes in the consumption-wealth ratio. In addition, due to contemporaneous correlations between stock returns and changes in the consumption-wealth ratio (its elements, such as consumption growth and asset growth), changes in the ratio may also capture current unexpected returns on stocks, $\varepsilon_t$. Thus, I regress changes in returns on changes in the consumption-wealth ratio as

$$
r_{t+1} - r_t = \alpha + \beta_1 \Delta c_t + \beta_2 \Delta a_t + \beta_3 \Delta y_t + u_{t+1}, \quad (11)
$$

where $u_{t+1}$ is white noise that is uncorrelated with the regressors. Based on the equation (6), I expect that $\beta_1 \approx \frac{1}{\rho_w^\lambda_1}$, $\beta_2 \approx \frac{1}{\rho_w}$, and $\beta_3 \approx \frac{1-\lambda}{\rho_w^\lambda}$, provided that all three regressors are significant.
However, if $\Delta c_t$ and $\Delta y_t$ are not sufficiently volatile, asset growth $\Delta a_t$ will be the main component of the consumption-wealth ratio that captures the variation of changes in expected returns on stocks. Thus, I also run the following regression using only aggregate asset growth,

$$r_{t+1} - r_t = \alpha + \beta \Delta a_t + u_{t+1}. \tag{12}$$

My hypothesis is that the slope coefficient is the reciprocal of investment ratio, $\beta \approx \frac{1}{\rho_w}$.

In addition, changes in the consumption-wealth ratio are also related with changes in returns more than one-period ahead as the equation (6) shows. Yet, by $\rho_w < 1$ and $\lambda < 1$, many of the terms in the RHS of (6) must be very small.\(^{14}\) Therefore, I predict changes in returns two-period ahead and add changes in returns as a control variable. Using only asset growth instead of changes in the consumption-wealth ratio, I run the following regression:

$$r_{t+1} - r_t = \alpha + \beta_4 \Delta a_{t-1} + \gamma(r_t - r_{t-1}) + u_{t+1}. \tag{13}$$

I suggest the above regressions based on the model in the equation (6). Looking at changes in returns is also useful when expected returns are highly persistent but realized returns have few serial correlations as follows.

Since autocorrelation of stock returns is

$$cov(r_{t+1}, r_t) = \phi \text{var}(\mu_t) + cov(\mu_t, \varepsilon_t),$$

stock returns have few serial correlations if expected returns are negatively related with unexpected returns with the magnitude of $cov(\mu_t, \varepsilon_t) \approx -\phi \text{var}(\mu_t)$. When expected stock returns are highly persistent, in particular, we have $cov(\mu_t, \varepsilon_t) \approx -\text{var}(\mu_t)$. In words, agents adjust expected returns upward when there are negative shocks to returns and downward when there are positive shocks so that returns appear uncorrelated over time. In this case, due to few autocorrelations, lagged stock returns may not be useful to predict returns. Instead, looking at first differences in returns may be an alternative since we may focus on shocks to expected returns as shown in the equation (10).

\(^{14}\) For instance, using $\rho_w = 0.3$ and $\lambda = 0.1$, $\lambda \rho_w^2 = 0.009$, $\lambda \rho_w^3 = 0.00081$ and so on while $\lambda \rho_w = 0.03$. I have $\rho_w = 0.3$ by regression results that propose investment ratio of around 0.28 and $\lambda = 0.1$ from Ludhig.
To test the equation (6), quarterly and annual real returns on value-weighted CRSP index are used to proxy asset returns \( (r_{a,t}) \). The household net worth data provided by the Board of Governors of the Federal Reserve is used for asset holdings \( (a_t) \). The net worth of households is calculated as total assets including real estate, durable goods (at replacement cost) and financial assets minus total liabilities such as home mortgages and consumer credit. The net worth used in the tests is in 2000 chain-weighted dollars with a Personal Consumption Expenditure (PCE) deflator and divided by population. The PCE deflator and population data are from National Income and Product Accounts (NIPAs). All values are in logarithms. The period is from the first quarter of 1952 to the fourth quarter of 2006 for quarterly returns and from 1952 to 2006 for annual returns.

Table 1 summarizes the descriptive statistics for the growth of aggregate asset holdings (net worth of households in the Flow of Funds) and equity returns (value-weighted CRSP index returns) for quarterly data. The sample mean and median of the logarithm of asset holdings, which are not stationary, are 11.26 and 11.21 respectively. These correspond to around $75,000 and $70,000 in 2000 dollars respectively. They have grown at 0.6\% in a quarter on average. Quarterly real equity returns are 1.8\% on average and 2.7\% in median while changes in the returns are 0.01\% on average. Moreover, changes in equity returns are highly volatile compared to the levels of returns (variance 0.013 vs. 0.007). Figure 1 shows the time series plots of asset growth and equity returns. Asset growth and changes in equity returns appear to be stationary.

The results of a serial correlation test for equity returns and asset growth are provided in Table 2. The tests show that we cannot reject that aggregate asset growth and equity returns have zero autocorrelations. For instance, quarterly stock returns have serial correlation coefficient of 0.03 with the standard error 0.067. Yet, the autocorrelation of changes in returns (-0.45) is less than -0.5. This suggests that stock returns are correlated over time (despite the statistical insignificance) and that expected returns are highly persistent (but stationary) as follows:

---

15 Financial assets include deposits, treasury securities, corporate equities, mutual fund shares and pension fund reserves. As of the fourth quarter of 2005, equity shares (directly and indirectly held) to total assets and to the net worth are 23.2\% and 38.2\% respectively (Table B.100.e of Flow of Funds provided by the Board of Governors of the Federal Reserve).

16 Lettau and Ludvigson (2001) also used a PCE deflator (1992=100) as a common deflator for consumption, asset holdings and labor income. The results provided here do not depend on which deflator is used.

17 The result improves slightly if we restrict the sample to the post-war period.
Using the equation (7), variance of changes in stock returns is

$$\text{var}(\Delta r_{t+1}) = 2\{(1 - \phi)\text{var}(\mu_t) + \text{var}(\varepsilon_{t+1}) - \text{cov}(\mu_t, \varepsilon_t)\},$$

(14)

and the serial correlation of changes in stock returns of order one is

$$\text{cov}(\Delta r_{t+1}, \Delta r_t) = (1 - \phi)\{\text{cov}(r_{t+1}, r_t) - \text{var}(\mu_t)\} - \{\text{var}(\varepsilon_{t+1}) - \text{cov}(\mu_t, \varepsilon_t)\}. $$

(15)

By the equation for $\text{var}(\Delta r_{t+1})$, we can rewrite (9) as

$$\text{cov}(\Delta r_{t+1}, \Delta r_t) = (1 - \phi)\text{cov}(r_{t+1}, r_t) - \frac{1}{2}\text{var}(\Delta r_{t+1}),$$

which leads to the autocorrelation coefficient of changes in returns:

$$\frac{\text{cov}(\Delta r_{t+1}, \Delta r_t)}{\text{var}(\Delta r_{t+1})} = -\frac{1}{2} + (1 - \phi)\frac{\text{cov}(r_{t+1}, r_t)}{\text{var}(\Delta r_{t+1})}. $$

(16)

In essence, when $(1 - \phi) = 0$ or $\text{cov}(r_{t+1}, r_t) = 0$ (or both), we must have the correlation coefficient equal to -0.5. Therefore, our sample estimates of -0.45 suggests that stock returns are correlated over time. Furthermore, we can obtain the persistency parameter $\phi$ by sample estimates of $\text{cov}(r_{t+1}, r_t)$ and $\text{var}(\Delta r_{t+1})$. After plugging in the estimates 0.03 and 0.013 respectively, I get the persistency parameter around 0.98. Therefore, over the quarterly horizons, expected stock returns appear highly persistent.

Correlations among the variables are noteworthy as shown in Table 1 (panel B). Although equities are not a major part of net worth of households, contemporaneous asset growth and equity returns are highly correlated, consistent with Stambaugh (1982, 1983). Equity returns are positively correlated with changes in equity returns while they are negatively correlated with lagged asset growth. Figure 1 shows the time series of lagged asset growth and changes in equity returns for quarterly data (bottom left) and annual data (bottom right).

For robustness and comparisons tests, we use a dividend-to-price ratio, consumption and labor income growths and “cay”. The dividend-to-price ratio of the CRSP value-weighted (VW) index is the ratio of dividends for the previous one year to the current value of the index. The dividends are calculated using the method adopted by Lettau and Nieuwerburgh (2007). Specifically, we get dividend yields by subtracting the quarterly returns on VW index without dividends from the quarterly returns with dividends. Then, dividends are obtained by multiplying the corresponding index value at the end of the previous quarter. The one-year dividends are the sum of the previous four-quarter dividends. For the variables consumption and labor income growths and “cay”, we use the updated quarterly data on Lettau’s website.
3.3 Results

Table 3 summarizes the results of OLS regressions to forecast changes in equity returns one quarter ahead. The dependent variable is changes in the value-weighted quarterly CRSP index returns, and the independent variables are a constant, lags of consumption growth, asset growth and labor income growth, and lags of changes in the returns depending on regression models. Rows for independent variables show the estimate of the coefficient, Newey-West corrected standard errors, t-value and p-value respectively. AdjRsqrt represents the adjusted R squared.

The equation (6) is tested by the last model (m12). The result shows that consumption growth and labor income growth do not play a role in predicting changes in equity returns. These variables are not statistically significant and their magnitudes are not economically important. In contrast, asset growth shows significant and important predictive power. The estimated slope of asset growth and adjusted R squared are virtually the same as those of the regression using only asset growth (m3). This confirms the assumptions that the return on human capital \( r_{h,t} \) and consumption growth \( \Delta c_t \) do not vary much from period to period and, thereby, proves my conjecture that asset growth alone may have forecasting power. Hence, I focus on the predictive ability of asset growth in the rest of the paper.

The equation (7) is tested by the model 3 (m3), whose result is summarized in

\[
\Delta r_{t+1} = \alpha + \beta \Delta a_t + u_{t+1} = 0.0224 - 3.5773 \Delta a_t.
\]  

(17)

Both the intercept and the coefficient of lag of asset growth show significant predictive power at the significance level of 1% with an adjusted R squared of around 39%. This R squared roughly corresponds to an R squared of around 3.5% when we regress the level of returns on the lag of asset growth (Appendix shows the relationship).

Moreover, the slope of aggregate asset growth has negative sign as predicted by the equation (7). Its value is -3.58, which implies that the steady-state level of investment to wealth ratio \( \rho_w \) is around 0.28. Figure 2 (top left) shows the plot of predicted changes in returns versus realized changes in returns when the independent variable is lagged asset growth.

Model 6 (m6) demonstrates regressions with an additional independent variable of lag of changes in returns. The result shows that it does not improve
the predictive power of aggregate asset growth to include one or two lags of
changes in returns. The regression is without an intercept since the regression
model 2 (m2) with lag of changes in returns shows that the intercept is almost
zero.

I test the equation (8) by the regression model 5 (m5). It shows that the sec-
ond lag of asset growth and the lag of change in returns can predict changes in
returns with less predictive power. Although the signs are correct, the coeffi-
cient level of the second lag of asset growth does not appear reasonable based
on the model’s predictions that are discussed in Section 2.2.

Lettau and Ludvigson document that “cay” predicts quarterly stock returns.
Provided that “cay” has the predictive ability for stock returns, first differences
of “cay” must predict changes in returns. Over the sample period of the first
quarter of 1952 to the fourth quarter of 2006, my regression of stock returns on
“cay” has the coefficient estimate of 1.6 with a t-stat of 3.85 and an adjusted
R squared of 6%. Given this predictive ability of “cay” for stock returns,
I regress changes in returns on first differences of “cay”. Yet, since changes
in “cay” are correlated with contemporaneous returns (-0.54, is significant),
the error terms after we regress changes in returns on changes in “cay” are
correlated with the regressor. Hence, I include lagged returns as a control
variable. The results show that the estimate of the slope coefficient for changes
in “cay” is around 1.3 with a t-stat of 1.39 while that for lagged returns is
-0.89 with a t-stat of -10.9. The adjusted R squared of the regression is 47%,
which is virtually the same as the R squared of the regression using only the
lagged returns. In essence, changes in “cay” appear to have little predictive
power for future changes in stock returns. Moreover, the predictive ability of
changes in “cay” and lagged returns in terms of a mean squared error is not
as good as that of asset growth (0.0108 vs. 0.008). Figure 2 (bottom right)
illustrates the predicted quarterly returns using changes in “cay” and lagged
returns. Their predicted values are virtually constant and comparable with
those values by asset growth (top right).

The results for annual returns are similar to those for quarterly returns de-
scribed above. As shown in Table 4, aggregate asset growth has significant
predictive ability at the 1% level with a coefficient estimate of -3.79, which is
very close to that for quarterly returns. The adjusted R squared is as high as
46%.

Predicted changes in returns may be used to predict future returns by adding

\[ y_{t+1} = \alpha + \beta x_t + \varepsilon_t \]

is a true model, \( \Delta y_{t+1} = \beta \Delta x_t + \varepsilon_t - \varepsilon_{t-1} \) must hold.

If we restrict the sample until the third quarter of 1998, we get an adjusted R
squared of around 9% after regressing on the “cay” and it increases slightly to 10%
if we add the lag of market returns to the independent variables—the same result
as those in Lettau and Ludvigson.
them to the current returns. Figure 2 (top right) shows realized returns against the predicted returns obtained by using only asset growth as a regressor. They are almost identical compared to those obtained from a regression of one-period ahead change in returns on asset growth and (current) change in returns (not shown in the figure). This implies that asset growth also have relevant information for unexpected returns, which (current) change in returns conveys. In essence, asset growth captures sufficient part of $v_t - \varepsilon_t$ in the equation (7). Indeed, as $v_t - \varepsilon_t$ is not serially correlated under the assumptions, asset growth is not (autocorrelation coefficient of -0.057, is insignificant).

On the other hand, the bottom left figure shows the predicted returns by “cay” after I regress returns on “cay”. The predicted values are almost constant compared with the predictions by asset growth (top right). In other words, the variation in fitted values by “cay” is less than the variation when using asset growth as a regressor. The variation is even less significant when I predict returns by obtaining fitted values of changes in returns by changes in “cay” and then adding them to lagged realized returns (bottom right figure).

4 Out-of-sample tests

An out-of-sample test is conducted and the results are described in Table 5. First, the slope of asset growth is estimated using the first half of the sample (from the first quarter of 1952 to the fourth quarter of 1979). This estimated coefficient, then, is used to estimate changes in expected returns in the other half period (from the first quarter of 1980 to the fourth quarter of 2006). Then we regress changes in realized returns in the second half period on these estimates of expected changes in returns (i.e., the fitted values). If the coefficient is close to one:

$$\Delta r_{t+1} = \beta \hat{E}_t[\Delta r_{t+1}] + u_{t+1}$$
$$\approx \hat{E}_t[\Delta r_{t+1}] + u_{t+1},$$

then we may conclude that our estimates of the expected changes in returns are unbiased forecasts of subsequent changes in realized returns out-of-sample as in

$$\hat{E}_t[\Delta r_{t+1}] \approx E_t[\Delta r_{t+1}],$$

where $\hat{E}_t[\Delta r_{t+1}]$ is the estimates of expected changes in returns.

The results for quarterly returns in Table 5 (panel A) show that the slope is significant and close to one, which suggests that the estimates of expected changes in quarterly returns are indeed unbiased predictors of realized changes in returns in an out-of-sample period. The adjusted R squared shows that the
variation in changes in returns can be explained by the variation in expected changes in returns using only the subsample as well as using the whole sample. However, for annual horizons, the fixed method is not appropriate to obtain out-of-sample estimates since sample size is too small as seen in the panel B. The coefficient is not close to one for annual returns.

On the other hand, we may use a recursive method to estimate expected changes in returns. For the first quarter of 1980, we use the slope of asset growth estimated in the first half-sample as described above. For the second quarter of 1980 and so on, however, we use the extended sample periods up to the previous quarter to re-estimate the slopes of asset growth. Then the fitted values of these regressions are used as independent variables for the regressions of changes in returns. The results, not reported, are almost the same as the panel A. This suggests that the first-half sample is sufficient to obtain consistent estimates of the slope of asset growth in quarterly data. The recursive method, however, improves the estimates for annual returns as seen in panel B and C.

With a large sample such as quarterly data, the slope of asset growth for changes in returns can be estimated consistently with a relatively short sample period. This provides unbiased predictors of subsequent realized changes in returns out of sample. The result is consistent with the theoretical model's implication that the slope should be the reciprocal of the steady-state level of the investment ratio. In practice, the powerful out-of-sample predictive power of asset growth is also useful. We do not need to re-estimate the slope for an extended sample period.

Moreover, consumption data or an estimation of the cointegrating relationship associated with consumption is not required to predict changes in returns because asset growth alone provides a reliable method to examine time variation in expected returns. The variable “cay” does not perform better out of sample than asset growth. Consumption growth and labor income growth appear to add little predictive ability. Since consumption data has measurement errors and the cointegrating relationship is subject to estimation errors, the univariate predictive power of asset growth is useful.

The results (not reported) show that recursive estimates of coefficients of “cay” change between 1 and 3 over the out-of-sample period. The mean squared error is 0.0117, which is larger than that of asset growth (0.008).
5 Robustness checks

5.1 Does asset growth predict future changes in stock returns?

Asset growth does explain variation in future changes in stock returns. Although stock returns and asset growth are correlated as high as 88% during the sample period (Table 1), it is not true that asset growth explains nothing but contemporaneous stock returns in the first differences of stock returns rather than future changes in stock returns. If that were the case, the regression should be similar with the regression using stock returns as the right hand side variable. However, the results are quite different.

To examine predictive ability of asset growth, we compare the regression using asset growth with the regression using stock returns. Stock returns have serial correlation close to zero in quarterly and annual horizons so they predict neither future stock returns nor future changes in stock returns. Under the null hypothesis that expected stock returns are constant, regressing changes in stock returns on current stock returns is, therefore, equivalent to regressing stock returns on a constant and getting historical averages as predicted values. This is also true under an alternative hypothesis that expected returns vary over time but are not correlated with current stock returns.

Table 6 shows that stock returns explain only contemporaneous returns, not future changes in returns (panel A, Null model). After regressing changes in returns on lagged stock returns, we get the coefficient estimate of around -0.95 with an adjusted $R^2$ squared around 47% in sample. The fitted values—predicted changes plus actual current returns—do not vary much as shown in panel C (Null model). The standard deviation is just 0.004, which is 5% of the standard deviation of actual returns (0.084). In other words, the predicted stock returns one period ahead using stock returns are virtually historical averages of stock returns as Figure 3 (top left) illustrates. The results are consistent with a hypothesis that expected returns are constant or not correlated with current stock returns.

Out-of-sample results do not differ from the in-sample results just described. In essence, the fitted values are almost constant, as in the case of the historical mean model. The out-of-sample test applies a recursive method. First, a slope of stock returns is estimated using the first half of the sample (from the first quarter of 1952 to the fourth quarter of 1979). Subsequently, we add one more observation at a time to estimate slopes of stock returns from the first quarter of 1980 to the fourth quarter of 2006. This procedure provides us with out-of-sample predicted values of changes in returns.

In contrast, the predicted future stock returns using asset growth exhibit a siz-
able variation according to a variation in asset growth (panel C, OLS model). Their standard deviation is around 50% of the standard deviation of realized returns. Maximum and minimum values differ significantly. While maximum predicted value by stock returns is just 3% in sample, the maximum obtained by asset growth is 15%, when actual returns are as high as 20%. Likewise, minimum predicted value by asset growth (-12%) is comparable to realized lowest return of -30%. Figure 5 illustrates the different patterns of predicted values by stock returns and asset growth out of sample (compare top figures).

5.2 Is the OLS regression robust to an endogeneity problem?

Although serial correlations in error terms may raise a concern about an endogeneity problem, the OLS estimator for asset growth appears consistent. A Hausman test does not reject the null hypothesis that the OLS and a two-stage least square estimates are statistically different. Moreover, the OLS estimate is not much different from the coefficient of fitted values of asset growth by an instrument variable (-3.57 vs. -3.71). Finally, out of sample, the OLS estimate leads to less noisy results than using the instrument variable.

Since asset growth and stock returns are highly, but not perfectly, correlated, it is not obvious if the OLS regressor suffers from an endogeneity problem. The correlation of 88% between asset growth and stock returns over the sample period implies that asset growth can capture a sizable part of contemporaneous stock returns. Indeed, the regression errors have a serial correlation of -0.1, which is much less than the autocorrelations of changes in returns (around -0.5). Although the size is small, autocorrelations in errors may imply that asset growth is correlated with error terms. Therefore, we examine whether this is significant enough to make the OLS estimator inconsistent. 21

To address this concern, we perform a Hausman test, which is often used to examine existence of endogeneity, using labor income growth as an instrument variable. 22 The Hausman test does not reject the null hypothesis that the OLS

21 When stock returns are not serially correlated, regressing changes in stock returns on stock returns gives zero correlation between the regressor and the regression errors (correlation of 0.002, is insignificant). Likewise, if asset growth were perfectly correlated with stock returns, there would be no endogeneity problem. Yet, in this case, asset growth would not have the predictive ability, either.

22 The labor income growth is uncorrelated with the regression errors. The correlation between labor income growth and contemporaneous stock returns is almost zero (0.07 with p value 0.30). Adding labor income growth is not helpful when we use asset growth to predict changes in returns (Table 3). In addition, asset and labor income growths are correlated (correlation coefficient 0.15 with p value 0.02). If we regress asset growth on labor income growth, F-value is around 5.10 (slope
estimator is consistent. The Chi-square statistic is only around 0.85, which represents a p-value of 0.65.

We also compare regression results using our instrument with the OLS results. The predictive regression procedure is as follows. We first regress asset growth on labor income growth to get fitted values. Then we use fitted values as a regressor for changes in stock returns. Since labor income growth is not correlated with stock returns, fitted values of asset growth from the first stage are not correlated with stock returns. This avoids the endogeneity problem but also causes serial correlations in errors that are comparable to changes in stock returns (-0.47 vs. -0.5). Therefore, to remove serial correlations in error terms, we can add stock returns as a control variable although it does not help us predict changes in returns. After adding stock returns, the error terms appear to be a white noise (serial correlation of -0.008, is insignificant).

As Table 6 shows, in sample (panel A), this procedure provides a coefficient estimate of asset growth over the whole sample period (-3.71), which is quite similar to the OLS estimate of -3.58. It leads to a slightly smaller mean squared error (0.007 vs. 0.008) and less volatile predicted returns than the OLS (compare Figure 3’s top right and bottom left). Out of sample, however, the instrument variable leads to noisier results while the OLS out of sample is not different from in sample. The coefficient estimate using the instrument is not stable over time (Figure 4, top left). This is because the first-stage regression coefficient—when we regress asset growth on labor income growth—varies over time. Especially, when the coefficient value is less than 1 as in our case, small changes in the first-stage coefficient have a large impact on the second-stage coefficient. This causes the second stage regression coefficient to vary. In addition, the instrument variable provides a larger mean squared error than the OLS (0.0118 vs. 0.0085) and much volatile predicted returns (compare Figure 5 top left and bottom right). These results confirm that the OLS estimate is consistent and using an instrument variable is not efficient.

6 Comparisons of out-of-sample predictive ability

We compare out-of-sample predictive ability of various variables—such as historical means, dividend-to-price ratio, “cay”, changes in “cay” and labor income growth—in terms of a mean squared predictive error as suggested by Clark and West (2007). The comparison indicates that the variable “cay”—

\[ 0.34 \text{ with t-value 2.26}. \]

\[ ^2 \text{Numerically, a two-stage least square estimate is the same as the ratio of an OLS estimate of regression of a dependent variable on an instrument to an OLS estimate of a regression of an independent variable on the instrument.} \]
estimated using the whole sample period—is the best over the out-of-sample period from the first quarter of 1980 to the fourth quarter of 2006. However, excluding the period when the markets were extremely volatile (from the third quarter of 2000 and the third quarter of 2002), the best predictor is aggregate asset growth. Moreover, in recent years (after the third quarter of 2002) asset growth has performed best.

To compare out-of-sample predictive ability of two models—a null model and an alternative that nests the null model—we can test if they have equal mean squared predictive errors (MSPEs). However, due to a size distortion, we cannot simply do a standard t-test for differences of MSPEs. As Clark and West show, the standard test results in a statistic that is not normally distributed. Rather, it is centered around a negative value in finite samples, which means that an alternative model has larger MSPE than the null model on average, when the null is true. The intuition behind this distortion is that an additional variable in an alternative model is not only unhelpful under the null but also harmful by adding noise to the predictive process. Therefore, in finite sample, the MSPE of an alternative model is expected to be larger than the MSPE of a null model. A simulation in Clark and West show that the size of a standard t-test is 3% for a 30-year quarterly period when a normal size is 10%. The size distortion becomes worse as we increase the out-of-sample periods; for example, it is 1% for a 60-year quarterly period.  

Clark and West suggest an adjustment to make the distribution approximate the standard normal distribution. The adjustment is to subtract out the noise that makes the MSPE with additional variables larger than the MSPE without the variables under the null. With the adjustment, size is improved: 9% and 7% for 30-year and 60-year quarterly out-of-sample periods respectively, as compared to the normal size of 10%. We, therefore, use their suggested t-statistics as described in Appendix C but also report t-statistics without the adjustment.

To perform the comparisons, we use various alternative models that nest a null model that expected stock returns are constant. Except for the variables asset growth and changes in “cay”, an alternative model regresses stock returns one period ahead on a constant and a predictive variable that is commonly used. In these cases, a null model is a regression using only a constant (historical mean model). On the other hand, when we use aggregate asset growth or changes in “cay” as an additional variable, only a constant and stock returns are dependent variables in a null model. Under the null hypothesis, changes in stock returns are a moving average process with an order of one and, thereby, only the component of contemporaneous stock returns can be explained by a contemporaneous variable. Under the alternative hypothesis that expected

\[24\] A recursive method is used to get the out-of-sample estimates.
returns vary over time and can be captured by asset growth, the alternative model uses a constant, stock returns and fitted values of asset growth by labor income growth.\footnote{Due to high correlations between asset growth and stock returns, we use fitted values of asset growth by labor income growth when we add stock returns as an independent variable.}

Table 7 shows the comparison results. The variable “cay”—constructed by the whole sample period—performs significantly better than the historical mean model over the whole out-of-sample period (t-stat of 2.42). Dividend-to-price ratio and labor income growth are not better than the historical mean model (t-stats are 0.71 and 0.51 respectively). Asset growth does not appear better than the null model. Its t-stat is 1.23, which implies that MSPEs between two models are not significantly different.

However, asset growth performs best if we exclude the early 2000’s when the stock markets were very volatile (from the third quarter of 2000 to the third quarter of 2002). The adjusted t-stat of “cay” becomes insignificant (1.89) but that of asset growth becomes significant (2.23). The other variables, however, do not improve. Both dividend-to-price ratio and labor income growth do not seem better than historical averages. Moreover, if we only look at recent quarters (17 quarters after the third quarter of 2002 until the fourth quarter of 2006), asset growth is still significantly better than the null model. Yet, the other variables do not beat historical means. Rather, dividend-to-price ratio, labor income growth and changes in “cay” have negative t-stats although most of them are not statistically significant. Changes in “cay” appears to be significantly worse than historical means in the recent periods. The poor performance may be attributed to high autocorrelations in the variable “cay”, which can cause more noise in the regression out of sample.

Last, Figure 6 shows out-of-sample predicted stock returns by the commonly-used predictive regressors. Obviously, the values are almost constant if we use historical mean model. For other variables such as “cay”, dividend-to-price ratio and labor income growth, the predicted stock returns do not vary much, compared to the regression using asset growth (Figure 5).

7 Conclusion

Although a standard view is that stock prices without a stationary component make it hard to predict stock returns, this paper shows that we can still predict stock returns by examining changes in expected returns and thereby predicting changes in stock returns. We find that the growth of aggregate asset holdings
is a univariate, powerful predictor of subsequent changes in equity returns. The changes in expected returns on stocks estimated by the asset growth explain a substantial portion of the variation in realized changes in returns. Moreover, the asset growth performs as well in an out-of-sample period as it does in the whole sample period. These empirical results are consistent with the relationship between changes in value of assets and changes in expected future returns (shocks to discount rates). Therefore, asset growth allows us to learn about the agent’s updates on expectations for future returns on assets, which are not directly observable.

We also show that asset growth predicts better out of sample in terms of a mean squared predictive error (Clark and West (2007)) than other predictors, such as the dividend-to-price ratio, “cay” and labor income growth. In particular, excluding the period when the markets were extremely volatile (from the third quarter of 2000 and the third quarter of 2002), only asset growth predicts better than the null model out of sample. These results suggest that using asset growth provides a convenient tool for predicting stock returns and enhances predictive ability out of sample.
References


[23] Rapach, David, Jack Strauss and Guofu Zhou, 2008, Diversification Also Works for Forecasting the Equity Premium: Consistently Outperforming the Historical Average, working paper.


Appendix

A. Consumption-wealth ratio

We start from the budget constraint

\[
\begin{align*}
W_{t+1} &= R_{w,t+1}(W_t - C_t) \\
\frac{W_{t+1}}{W_t} &= R_{w,t+1}(1 - \frac{C_t}{W_t})
\end{align*}
\]  (A-1) (A-2)

where \(W_t\) is the aggregate wealth in period \(t\), \(R_{w,t+1}\) is the gross return on aggregate wealth and \(C_t\) is the consumption. If we take log to the equation (A-2) and use lower case letters to denote value in log,

\[
w_{t+1} - w_t = r_{w,t+1} + \log(1 - \exp(c_t - w_t)).
\]  (A-3)

Assuming log consumption-wealth ratio does not vary much, we can approximate the last element of RHS of (A-3) by a first Taylor expansion around a stationary level \(c - w = \log(\frac{C}{W})\), following Campbell (1993), as

\[
\log(1 - \exp(c_t - w_t)) \approx \frac{-\exp(c - w)}{1 - \exp(c - w)} (c_t - w_t) - (c - w)).
\]  (A-4)

Let us \(k\) represent the remaining constant terms in RHS of (4) and \(\rho_w\) the steady-state investment to wealth ratio, \(\frac{W-C}{W}\). This allows us to express as

\[
\log(1 - \exp(c_t - w_t)) \approx \frac{-\frac{C}{W}}{1 - \frac{C}{W}} (c_t - w_t) + k
\]

\[
\approx (1 - \frac{1}{\rho_w})(c_t - w_t) + k.
\]  (A-5)

Then we can approximate (A-3) using (A-5) and then solve forward the first differential equation:
\[ w_{t+1} - w_t \approx r_{w,t+1} + (1 - \frac{1}{\rho_w})(c_t - w_t) + k \]

\[
\begin{align*}
\rho_w (w_{t+1} - w_t) &\approx \rho_w r_{w,t+1} + (\rho_w - 1)(c_t - w_t) + \rho_w k \\
ct - w_t &\approx \rho_w r_{w,t+1} - \rho_w (w_{t+1} - w_t) + \rho_w (c_t - w_t) + \rho_w k \\
ct - w_t &\approx \rho_w r_{w,t+1} + \rho_w (c_{t+1} - w_{t+1} - c_{t+1} + c_t - c_t + w_t) + \rho_w (c_t - w_t) + \rho_w k
\end{align*}
\]

where \( c_{t+1} - w_{t+1} \approx \rho_w r_{w,t+2} + \rho_w (c_{t+2} - w_{t+2}) - \rho_w (c_t - w_t) + \rho_w k \)

\[
c_t - w_t \approx \sum_{i=1}^{\infty} \rho_w^i (r_{w,t+i} - \Delta c_{t+i}) + \lim_{n \to \infty} \rho_w^n (c_{t+n} - w_{t+n}) + \sum_{i=1}^{\infty} \rho_w^i. \tag{A-6}
\]

Finally if we use \( \lim_{n \to \infty} \rho_w^n (c_{t+n} - w_{t+n}) = 0 \) and omit the constant term for simplicity in the RHS of (A-6), we can get

\[
c_t - w_t \approx \sum_{i=1}^{\infty} \rho_w^i (r_{w,t+i} - \Delta c_{t+i}). \tag{A-7}
\]

As (A-7) must hold ex-post, we can also take conditional expectation on information at time \( t \) on both sides to get

\[
c_t - w_t \approx E_t \left[ \sum_{i=1}^{\infty} \rho_w^i (r_{w,t+i} - \Delta c_{t+i}) \right] \tag{A-8}
\]

as desired.

The equation (A-8) implies that current consumption-wealth ratio summarizes expectation of future discounted wealth returns net of consumption growth, conditional on current information. Therefore, if consumption growth does not vary much, current consumption wealth ratio can be a good proxy for expectations of future returns on wealth.

7.1 B.

Assume that aggregate wealth at time \( t \) (\( W_t \)) comprises asset holdings (\( A_t \)) and human capital (\( H_t \)) and that both are tradable. Let us denote a variable in logarithm by a lower case letter. For example, \( w_t \) is the logarithm of \( W_t \), \( r_{w,t+i} \) is the logarithm of the gross return on wealth for one period from \( t+i-1 \) to \( t+i \), and \( c_t \) is the logarithm of consumption \( C_t \) in the period \( t \). As proved
in the appendix, \( c_t - w_t \), the logarithm of the consumption-wealth ratio \( \left( \frac{C_t}{W_t} \right) \), may be approximated as

\[
c_t - w_t \approx E_t \left[ \sum_{i=1}^{\infty} \rho_w^i (r_{w,t+i} - \Delta c_{t+i}) \right]
\]

(18)

where \( E_t \) denotes the expectation conditional on information in the period \( t \) and \( \rho_w \) is the steady-state level of investment to aggregate wealth \( \left( \frac{W_t - C_t}{W_t - W_0} \right) \).

Note that the consumption-wealth ratio cannot be measured since human capital \( (H_t) \) is not observable. Instead, we first use the identities

\[
W_t = A_t + H_t = \exp(a_t) + \exp(h_t)
\]

\[
w_t = \log(\exp(a_t) + \exp(h_t))
\]

(19)

and then approximate \( w_t \) around \( w_{t-1} \) using a first-order Taylor approximation.\(^{26}\) This gives us

\[
w_t \approx w_{t-1} + \frac{\exp(a_{t-1})}{\exp(a_{t-1}) + \exp(h_{t-1})} \Delta a_t + \frac{\exp(h_{t-1})}{\exp(a_{t-1}) + \exp(h_{t-1})} \Delta h_t
\]

\[
\Delta w_t \approx \frac{A_{t-1}}{W_{t-1}} \Delta a_t + \frac{H_{t-1}}{W_{t-1}} \Delta h_t.
\]

(20)

If we assume the asset holdings to wealth ratio \( \left( \frac{A_{t-1}}{W_{t-1}} \right) \) does not vary much from the steady-state level \( \left( \frac{A_t}{W_t} \right) \), we can replace the ratio in (3) with the steady-state level, which is denoted by \( \lambda \), to obtain\(^{27}\)

\[
\Delta w_t \approx \lambda \Delta a_t + (1 - \lambda) \Delta h_t.
\]

(21)

As Lettau and Ludvigson argue, human capital \( (h_t) \) may be decomposed into

\(^{26}\) Using a second-order Taylor approximation around \( w_{t-1} \) introduces additional terms including \( \Delta a_t^2 \) and \( \Delta h_t^2 \). However, regression results (not tabulated) show that these higher order terms are not statistically significant. Note that the equation (3) holds exactly for discretely compounded returns.

\(^{27}\) Lettau and Ludvigson (2001) approximate the logarithms of the levels by \( w_t \approx \omega a_t + (1 - \omega) h_t \). The approximation error can be seen from the fact that wealth \( (W_t) \) is the sum of asset holdings \( (A_t) \) and human capital \( (H_t) \) and thereby always greater than maximum of the two components. Therefore, the convex combination of the logarithms of asset holdings and the logarithm of human capital is not a proper approximation of the logarithm of aggregate wealth. If we approximate around the mean of \( w_t \), a convex combination is valid with a constant term. Yet, since \( w_t \) is nonstationary (its mean does not exist), this approach also results in a poor approximation.
a nonstationary component and some stationary process \( z_t \). In particular, if we assume the nonstationary component is well captured by aggregate labor income \((y_t)\), we can express \( h_t = k + y_t + z_t \) where \( k \) is a constant.

If we further regard labor income as the dividends of human capital as do Campbell (1996) and Jagannathan and Wang (1996), the return on human capital is given by \( R_{h,t+1} = \frac{H_{t+1} + Y_{t+1}}{H_t} \) where \( Y_t \) is labor income.\(^{28}\) The appendix shows that we can express

\[
 z_t = v + E_t \left[ \sum_{i=0}^{\infty} \rho_h^i (\Delta y_{t+1+i} - r_{h,t+1+i}) \right],
\]

where \( v \) is a constant and \( \rho_h = \frac{H}{H + Y} \) is the steady-state level of the human capital ratio.

Since \( \Delta h_t \) is the sum of labor income growth \((\Delta y_t)\) and the first difference of the stationary component \((\Delta z_t)\), \((4)\) can be rewritten as\(^{29}\)

\[
 \Delta w_t \approx \lambda \Delta a_t + (1 - \lambda)(\Delta y_t + \Delta z_t).
\]

This approximation for wealth growth can be used to express \((1)\) since \( c_t - w_t \) is previous period’s consumption wealth ratio plus the consumption growth and the wealth growth,

\[
c_{t-1} - w_{t-1} + \Delta c_t - \Delta w_t \approx E_t \left[ \sum_{i=1}^{\infty} \rho_w^i (r_{w,t+i} - \Delta c_{t+i}) \right],
\]

and by \((6)\),

\[
c_{t-1} - w_{t-1} + \Delta c_t - (\lambda \Delta a_t + (1 - \lambda)(\Delta y_t + \Delta z_t)) \approx E_t \left[ \sum_{i=1}^{\infty} \rho_w^i (r_{w,t+i} - \Delta c_{t+i}) \right].
\]

Notice that the consumption-wealth ratio in the previous period \((c_{t-1} - w_{t-1})\) can be expressed similarly to the equation \((1)\). Then the equation \((8)\) becomes

\(^{28}\) This enables us to proxy return on human capital by labor income growth.
\(^{29}\) If we use a second-order Taylor approximation around \( w_{t-1} \) instead of \((3)\), we have an analogue to the equation \((6)\),

\[
 \Delta w_t \approx \lambda \Delta a_t + (1 - \lambda)(\Delta y_t + \Delta z_t) + \frac{1}{2} \lambda (1 - \lambda)(\Delta a_t - \Delta y_t - \Delta z_t)^2.
\]

Yet, the regression results (not reported) suggest that the high-order terms such as \( \Delta a_t^2, \Delta y_t^2, \), and \( \Delta a_t \Delta y_t \) are not statistically significant.
\[
\Delta c_t - (\lambda \Delta a_t + (1 - \lambda)(\Delta y_t + \Delta z_t)) \\
\approx E_t \left[ \sum_{i=1}^{\infty} \rho_w^i (r_{w,t+i} - \Delta c_{t+i}) \right] - E_{t-1} \left[ \sum_{i=1}^{\infty} \rho_w^i (r_{w,t-1+i} - \Delta c_{t-1+i}) \right]. \tag{26}
\]

Campbell (1996) shows the approximation of the return on wealth, \( r_{w,t+i} \approx \lambda r_{a,t+i} + (1 - \lambda) r_{h,t+i} \). This allows us to approximate the RHS of the equation (9), leading to:

\[
\Delta c_t - (\lambda \Delta a_t + (1 - \lambda) \Delta y_t) \\
\approx E_t \left[ \sum_{i=1}^{\infty} \rho_w^i (\lambda r_{a,t+i} + (1 - \lambda) r_{h,t+i} - \Delta c_{t+i}) \right] - \\
E_{t-1} \left[ \sum_{i=1}^{\infty} \rho_w^i (\lambda r_{a,t-1+i} + (1 - \lambda) r_{h,t-1+i} - \Delta c_{t-1+i}) \right] + (1 - \lambda) \Delta z_t. \tag{27}
\]

As employed by Lettau and Ludvigson, if consumption growth \( (\Delta c_t) \) and the return on human capital \( (r_{h,t}) \) do not vary much from period to period, the conditional expectations of them vary even less and thus we may assume

\[ E_t[r_{h,t+i}] - E_{t-1}[r_{h,t-1+i}] \approx 0 \quad \text{and} \quad E_t[\Delta c_{t+i}] - E_{t-1}[\Delta c_{t-1+i}] \approx 0. \quad \tag{30}\]

Moreover, if \( r_{h,t} \) does not vary much, labor income growth \( (\Delta y_t) \) does not change much either since labor income is dividends on human capital as we assumed earlier. Then we also have \( E_t[\Delta y_{t+i}] - E_{t-1}[\Delta y_{t-1+i}] \approx 0 \). This leads to \( \Delta z_t = 0 \) by (5) since \( \Delta z_t = E_t[\sum_{i=0}^{\infty} \rho_h^i (\Delta y_{t+i+1} - r_{h,t+1+i})] - E_{t-1}[\sum_{i=0}^{\infty} \rho_h^i (\Delta y_{t+i} - r_{h,t+i})] \). As a result, we can further simplify the expression (10) to arrive at

\[
\Delta c_t - (\lambda \Delta a_t + (1 - \lambda) \Delta y_t) \approx \sum_{i=1}^{\infty} \rho_w^i \lambda \{ E_t[r_{a,t+i}] - E_{t-1}[r_{a,t-1+i}] \}. \tag{28}
\]

Notice that by \( \rho_w < 1 \) and \( \lambda < 1 \), most of the terms in the RHS of (11) must be very small since they are weighted by \( \rho_w^i \lambda \). Therefore, we may ignore all

\[ \text{Using quarterly data from the fourth quarter of 1951 to the fourth quarter of 2005, consumption growth has mean 0.0051 and standard deviation 0.0045. Labor income growth (proxy for return on human capital) has the sample mean 0.0060 and the standard deviation 0.0089. Alternatively, if we assume} \]

\[ c_t + E_t \sum_{i=1}^{\infty} \rho_w^i \Delta c_{t+i} \approx c_{t-1} + E_{t-1} \sum_{i=1}^{\infty} \rho_w^i \Delta c_{t-1+i}, \]

it is straightforward to arrive at the equation (13), provided that \( \Delta y_t \) does not have much information about changes in expected returns.
the terms except for the first one and approximate the RHS as

\[ \Delta c_t - (\lambda \Delta a_t + (1 - \lambda) \Delta y_t) \approx \rho_w \lambda \left\{ E_t[r_{a,t+1}] - E_{t-1}[r_{a,t}] \right\}. \quad (29) \]

The equation (12) suggests that the LHS—the change in the variable “cay” constructed by Lettau and Ludvigson—has information about changes in expected returns. However, if \( \Delta c_t \) and \( \Delta y_t \) do not vary much, they contain little information about the change in expected returns. Therefore, only the asset growth (\( \Delta a_t \)) may have relevant information about changes in expected returns. As a result, we may estimate changes in expected returns by

\[ -\Delta a_t + k \approx \rho_w \left\{ E_t[r_{a,t+1}] - E_{t-1}[r_{a,t}] \right\} \quad (30) \]

or if we rearrange,

\[ E_t[r_{a,t+1}] - E_{t-1}[r_{a,t}] \approx -\frac{1}{\rho_w} \Delta a_t + \frac{k}{\rho_w} \approx -\frac{1}{\rho_w} \Delta a_t + \alpha \quad (31) \]

where \( k \equiv \frac{1}{\lambda} \{ \Delta c_t - (1 - \lambda) \Delta y_t \}, \alpha \equiv \frac{k}{\rho_w}. \) In descriptive terms, then, we may say that changes in expected return are negatively related to growth of aggregate asset holdings and the coefficient of the asset growth is the reciprocal of the steady-state investment ratio in absolute value. This is consistent with Campbell and Shiller (1988) and Campbell (1991), who suggest that an increase in expected future returns on an asset is related to a decrease in value of the asset today. In other words, current value of aggregate asset holdings decreases if the expected returns one-period ahead increase.

On the other hand, if we don’t ignore the second term in the RHS of (11), then the following would hold true:

\[
\begin{align*}
\Delta c_t - (\lambda \Delta a_t &+ (1 - \lambda) \Delta y_t) \\
&\approx \rho_w \lambda \left\{ E_t[r_{a,t+1}] - E_{t-1}[r_{a,t}] \right\} + \rho_w^2 \lambda \left\{ E_t[r_{a,t+2}] - E_{t-1}[r_{a,t+1}] \right\}. 
\end{align*}
\]

If \( \Delta c_t \) and \( \Delta y_t \) do not vary much, we have

\[ -\Delta a_t + k \approx \rho_w \left\{ E_t[r_{a,t+1}] - E_{t-1}[r_{a,t}] \right\} + \rho_w^2 \left\{ E_t[r_{a,t+2}] - E_{t-1}[r_{a,t+1}] \right\} \]

or after rearranging

\[ E_t[r_{a,t+1}] - E_{t-1}[r_{a,t}] \approx -\frac{1}{\rho_w} \Delta a_t + \frac{k}{\rho_w} + \rho_w^2 \lambda \left\{ E_t[r_{a,t+2}] - E_{t-1}[r_{a,t+1}] \right\}. \]

\[ \text{As shown in Section 4, the regression results confirm that the consumption growth and the labor income growth do not add predictive power to the asset growth. In other words, the equation (14) does not improve by allowing variations in the consumption growth and the labor income growth.} \]

30
\[ E_t[r_{a,t+2}] - E_{t-1}[r_{a,t+1}] \approx \frac{k}{\rho_w^2} - \frac{1}{\rho_w} \{ E_t[r_{a,t+1}] - E_{t-1}[r_{a,t}] \} - \frac{1}{\rho_w^2} \Delta a_t \]

\approx \alpha_2 - \frac{1}{\rho_w} \{ E_t[r_{a,t+1}] - E_{t-1}[r_{a,t}] \} - \frac{1}{\rho_w^2} \Delta a_t \quad (32)\]

where \( k \equiv \frac{1}{3} \{ \Delta c_t - (1 - \lambda)\Delta y_t \} \) and \( \alpha_2 \equiv \frac{k}{\rho_w^2} \). The equation (15) suggests that we may estimate the change in expectation of returns two-periods ahead by using the current asset growth and the change in expectation of returns in the next period.

In general, allowing the \( n \) number of terms in the equation (11) gives us the change in expectations of returns \( n \)-periods ahead:

\[ E_t[r_{a,t+n}] - E_{t-1}[r_{a,t+n-1}] \approx \alpha_n - \sum_{i=1}^{n-1} \frac{1}{\rho_{n-i}} \{ E_t[r_{a,t+i}] - E_{t-1}[r_{a,t+i-1}] \} - \frac{1}{\rho_w^2} \Delta a_t \]

where \( \alpha_n \equiv \frac{k}{\rho_n^2} \) and \( k \equiv \frac{1}{3} \{ \Delta c_t - (1 - \lambda)\Delta y_t \} \).

The above relationships between changes in expectation and asset growth have empirical implications to test. We can substitute the change in expectation \( (E_t[r_{a,t+1}] - E_{t-1}[r_{a,t}]) \) in the equation (14) with the change in returns \( (\Delta r_{a,t+1}) \) using

\[ r_{a,t+1} = E_t[r_{a,t+1}] + \varepsilon_{t+1} \quad (33) \]

where \( \varepsilon_{t+1} \) is white noise with \( E_t[\varepsilon_{t+1}] = 0 \). The substitution yields

\[ \Delta r_{a,t+1} \approx \alpha - \frac{1}{\rho_w} \Delta a_t + \delta_{t+1} \quad (34) \]

where \( \delta_{t+1} \) is a moving average process of order one.\(^{32}\)

On the other hand, the equation (15) is not straightforward since the change in expectations of returns two periods ahead may not be substituted using the equation (16) unless there is no new information between the current period and the next period. Nevertheless, if we substitute, assuming the change in expectations of returns two period ahead is not too different from the change in expectations of returns one period ahead, we have

\[ \Delta r_{a,t+2} \approx \alpha' - \frac{1}{\rho_w} \Delta r_{a,t+1} - \frac{1}{\rho_w^2} \Delta a_t + \psi_{t+2}. \quad (35) \]

\(^{32}\)When asset growth is correlated with contemporaneous stock returns, the coefficient of the MA (1) process \( \delta_{t+1} \) has absolute value less than 1. Over the sample period of 1952 to 2006, the estimate for the first order autocorrelation of \( \delta_{t+1} \) is -0.1. Thus, the coefficient of the MA (1) is around -0.1 since the first order autocorrelation is equal to \( \frac{\theta}{1-\theta} \) if we denote the MA(1) coefficient by \( \theta \).
where $\psi_{t+2} = \delta_{t+2} + \frac{1}{\rho_w} \delta_{t+1}$. Obviously, $\psi_{t+2}$ is correlated with $\psi_{t+1} (= \delta_{t+1} + \frac{1}{\rho_w} \delta_{t})$ and $\psi_{t} (= \delta_{t} + \frac{1}{\rho_w} \delta_{t-1})$ since $\delta_{t+1}$ is correlated with $\delta_{t}$. Therefore, the error terms have the serial correlation of the order two.

For comparison, the equation (19) shows the model in Lettau and Ludvigson (2001),

\[ c_t - (\lambda a_t + (1 - \lambda) y_t) \approx E_t \left[ \sum_{i=1}^{\infty} \rho_w^i (\lambda r_{a,t+i} + (1 - \lambda) r_{h,t+i} - \Delta c_{t+i}) \right] + (1 - \lambda) z_t, \]

which suggests that the deviation from the cointegrating relationship among consumption, asset holdings and labor income can predict future asset returns provided that expected future returns on human capital ($r_{h,t}$) and consumption growth ($\Delta c_t$) do not vary much. However, this predictive relationship is derived by approximating wealth as a convex combination of its two components, asset holdings and human capital. While the return on wealth is a convex combination, the level of wealth is the sum of the two components and its level must exceed that of each component. The corrected model is suggested by the equation (10). Furthermore, if we apply the same assumptions such as on consumption and the returns on human capital as employed by Lettau and Ludvigson, we arrive at (14). The equation (14) does not require cointegration estimates to examine time variation in expected returns.

C. Stationary component of human capital

If we consider labor income ($Y$) as dividend of human capital,

\[ R_{h,t+1} H_t = H_{t+1} + Y_{t+1}, \]

where $R_{h,t+1}$ is the gross return on human capital $H_t$.

Dividing both sides by $Y_{t+1}$ and then taking log gives

\[ \frac{R_{h,t+1} H_t}{Y_{t+1}} = \frac{H_{t+1}}{Y_{t+1}} + 1 \]

\[ r_{h,t+1} + h_t - y_{t+1} = \log \left( \frac{H_{t+1}}{Y_{t+1}} + 1 \right) \]

\[ = \log \left( \exp (h_t - y_{t+1}) + 1 \right). \]

By decomposing $h_t$ as constant $k$ plus nonstationary component captured by labor income $y_t$ and some stationary component $z_t$, we have
\[ r_{h,t+1} + k + y_t - y_{t+1} + z_t = \log(\exp(h_{t+1} - y_{t+1}) + 1). \]  \hfill (A-9)

If we use Taylor approximation around steady-state level of human capital to labor income ratio, \( \bar{h} - \bar{y} \), to (A-9),

\[
\begin{align*}
    r_{h,t+1} + k + y_t - y_{t+1} + z_t &\approx \log(\exp(\bar{h} - \bar{y}) + 1) + \frac{\exp(\bar{h} - \bar{y})}{\exp(h - y) + 1} (h_{t+1} - y_{t+1} - (\bar{h} - \bar{y})) \\
    &\approx b + \frac{\exp(\bar{h} - \bar{y})}{\exp(h - y) + 1} (h_{t+1} - y_{t+1}) \\
    &\approx b + \frac{\exp(\bar{h} - \bar{y})}{\exp(h - y) + 1} (k + z_{t+1})
\end{align*}
\]

where \( b \) is the remaining constant. Define \( \frac{\exp(\bar{h} - \bar{y})}{\exp(h - y) + 1} = \frac{\bar{y}}{\bar{x}+1} = H = \rho_h \). We have the approximate expression for \( z_t \):

\[
\begin{align*}
    z_t &\approx -(r_{h,t+1} + k - \Delta y_{t+1}) + b + \rho_h (k + z_{t+1}) \\
    &\approx -(r_{h,t+1} + k - \Delta y_{t+1}) + b + \rho_h (k - (r_{h,t+2} + k - \Delta y_{t+2}) + b + \rho_h (k + z_{t+2})) \\
    &\approx -(r_{h,t+1} + k - \Delta y_{t+1}) + b + \rho_h (-r_{h,t+2} - \Delta y_{t+2}) + b + \rho_h^2 (k + z_{t+2})) \\
    &\approx \sum_{j=0}^{\infty} \rho_h^j (-r_{h,t+1+j} + \Delta y_{t+1+j}) + v
\end{align*}
\]

where \( v \) is the remaining constant, assuming \( \lim_{n \to \infty} \rho_h^j k = 0 \).

Since this should hold at time \( t \), taking conditional expectation at time \( t \) on the RHS gives us the equation (5) in the section 1 without the constant (we may ignore the constant since we only need \( \Delta z_t = 0 \) for the equation (11) in the section 1).

D. R squared

If we regress changes in returns, \( r_{t+1} - r_t \), on \( \Delta a_t \), we get the R squared,

\[
R^2 = \frac{\text{Var}(\beta \Delta a_t)}{\text{Var}(r_{t+1} - r_t)} \\
= \frac{\text{cov}(r_{t+1} - r_t, \Delta a_t)^2}{\text{var}(r_{t+1} - r_t) \text{var}(\Delta a_t)}.
\]
If $r_t$ has few serial correlations and we ignore them, the above R squared becomes

$$R^2 = \frac{\{\text{cov}(r_{t+1}, \Delta a_t) - \text{cov}(r_t, \Delta a_t)\}^2}{2 \text{var}(r_{t+1}) \text{var}(\Delta a_t)}$$

$$= \frac{\text{cov}(r_{t+1}, \Delta a_t)^2 - 2 \text{cov}(r_{t+1}, \Delta a_t) \text{cov}(r_t, \Delta a_t) + \text{cov}(r_t, \Delta a_t)^2}{2 \text{var}(r_{t+1}) \text{var}(\Delta a_t)}$$

$$= \frac{\text{cov}(r_{t+1}, \Delta a_t)^2 - \text{cov}(r_{t+1}, \Delta a_t) \text{cov}(r_t, \Delta a_t)}{2 \text{var}(r_{t+1}) \text{var}(\Delta a_t)} + \frac{\text{cov}(r_t, \Delta a_t)^2}{2 \text{var}(r_{t+1}) \text{var}(\Delta a_t)}.$$

Now define the correlation coefficient between the contemporaneous equity returns and asset growth, and that of the equity returns and lag of asset growth by $\rho_0$ and $\rho_1$ respectively. Then we can express as

$$\frac{\text{cov}(r_{t+1}, \Delta a_t) \text{cov}(r_t, \Delta a_t)}{\text{var}(r_{t+1}) \text{var}(\Delta a_t)} = \frac{\text{cov}(r_{t+1}, \Delta a_t)}{\sqrt{\text{var}(r_{t+1}) \text{var}(\Delta a_t)}} \frac{\text{cov}(r_t, \Delta a_t)}{\sqrt{\text{var}(r_t) \text{var}(\Delta a_t)}} = \rho_1 \rho_0,$$

and

$$\frac{\text{cov}(r_t, \Delta a_t)^2}{\text{var}(r_{t+1}) \text{var}(\Delta a_t)} = \left(\frac{\text{cov}(r_t, \Delta a_t)}{\sqrt{\text{var}(r_t) \text{var}(\Delta a_t)}}\right)^2 = \rho_0^2.$$

These lead to

$$R^2 = \frac{1}{2} R^*^2 - \rho_1 \rho_0 + \frac{1}{2} \rho_0^2,$$

where $R^*^2$ is the R squared when we regress $r_{t+1}$ on $\Delta a_t$. Thus, we have the $R^*^2$ as given by

$$R^*^2 = 2R^2 - (\rho_0^2 - 2\rho_1 \rho_0).$$

Using $\hat{\rho}_0 = 0.8803$ and $\hat{\rho}_1 = 0.0175$ over the sample period from 1952 to 2006, we have

$$R^*^2 = 2R^2 - 0.7441.$$

D. Clark and West (2006, 2007) adjustment

Consider a prediction for $y$ for an out-of-sample period $P$ from $T - P + 2$ to $T + 1$ using a rolling estimate method. Define the out of sample mean squared prediction (MSPE) as

$$\hat{\sigma}^2 = \frac{1}{P} \sum_{t=T-P+1}^{T} e_{t+1}^2,$$

where $e_{t+1} = y_{t+1} - \hat{y}_{t+1}.$
We may test equal mean squared prediction error (MSPE) between two models (null and alternative) out of sample. Clark and West (2006, 2007), however, alarms for the use of a standard t-test for the difference between two MSPE,

\[
\hat{\sigma}_0^2 - \hat{\sigma}_1^2 = \frac{1}{P} \sum_{t=T-P+1}^{T} e_{0,t+1}^2 - \frac{1}{P} \sum_{t=T-P+1}^{T} e_{1,t+1}^2 ,
\]

where \( e_{0,t+1}^2 \) is the squared error of a null model and \( e_{0,t+1}^2 \) is the squared error of an alternative model. Clark and West show that, in many cases, a t-test for this difference results in statistics whose distribution is centered around a negative value. This distortion in finite sample increases as an out-of-sample period \( P \) increases. Therefore, Clark and West suggest to test if

\[
\frac{1}{P} \sum_{t=T-P+1}^{T} 2e_{0,t+1}(e_{0,t+1} - e_{1,t+1})
\]

is zero.

This adjusted difference of MSPE is based on the observation that, in finite sample, \( \hat{\sigma}_0^2 - \hat{\sigma}_1^2 \), is likely to be negative because additional variables of an alternative model introduce noise in a prediction process.

Consider a simple example. A null model is that the variable \( y \) is white noise while an alternative model states that it can be predicted using predictors \( X \). The difference between MSPE is, then,

\[
\hat{\sigma}_2^2 - \hat{\sigma}_1^2 = \frac{1}{P} \sum_{t=T-P+1}^{T} (y_{t+1})^2 - \frac{1}{P} \sum_{t=T-P+1}^{T} (y_{t+1} - X_t' \hat{\beta}_t)^2 \\
= \frac{1}{P} \sum_{t=T-P+1}^{T} (y_{t+1})^2 - \frac{1}{P} \sum_{t=T-P+1}^{T} \left\{ (y_{t+1})^2 - 2y_{t+1}X_t' \hat{\beta}_t + (X_t' \hat{\beta}_t)^2 \right\} \\
= \frac{1}{P} \sum_{t=T-P+1}^{T} 2y_{t+1}X_t' \hat{\beta}_t - \frac{1}{P} \sum_{t=T-P+1}^{T} (X_t' \hat{\beta}_t)^2 . \tag{A-12}
\]

Under the null that \( y_{t+1} = \epsilon_{t+1} \), we expect \( E[y_{t+1}X_t' \hat{\beta}_t] = E[\epsilon_{t+1}X_t' \hat{\beta}_t] = 0 \) since \( \epsilon_{t+1} \) is not correlated with \( X \). Then the first term in the equation (A-12) is \( \frac{1}{P} \sum_{t=T-P+1}^{T} 2y_{t+1}X_t' \hat{\beta}_t \approx 0 \), in finite sample by the law of large numbers. However, the second term may not be zero but negative even though the null is true due to noise in finite sample. Therefore, we may subtract the second term and then test if the following adjusted difference is zero:
\[
\hat{\sigma}_2^2 - \hat{\sigma}_1^2 - \text{adjusted} = \frac{1}{P} \sum_{t=T-P+1}^{T} 2y_{t+1}X_t'\hat{\beta}_t \\
= \frac{1}{P} \sum_{t=T-P+1}^{T} 2y_{t+1} \left\{ y_{t+1} - (y_{t+1} - X_t'\hat{\beta}_t) \right\} \\
= \frac{1}{P} \sum_{t=T-P+1}^{T} 2e_{0,t+1}(e_{0,t+1} - e_{1,t+1}). \quad (A-13)
\]

When the null model is more general, Clark and West (2007) show that a similar argument also applies and suggest to do a t-test for the equation (A-13).