Dynamic Posted Pricing Scheme: Existence and Uniqueness of Equilibrium Bidding Strategy

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Abstract

We scrutinize the uniqueness issue of the equilibrium behavior of strategic customers making inter-temporal purchase decisions. We present that multiple equilibria can exist even in the simple setting where two identical customers compete for one unit of item. We prove the existence of a unique equilibrium when their valuations follow uniform distribution.

1 Introduction

We study the equilibrium behavior of strategic customers who make inter-temporal purchasing decisions in the presence of a markdown dynamic pricing scheme. Dynamic pricing schemes are prevalently used in practice – ranging from the simple markdown pricing in the fashion or retail industry to more complicated pricing schemes in the airline industry. At the same time, customers have also become strategic in their decision making as they factor in several issues into consideration, such as the timing of the purchase, future price, and in-stock probability. Understanding strategic customer behavior becomes a critical part for firms to design dynamic pricing schemes. See, for example, Talluri and Van Ryzin (2005), Shen and Su (2007) and Aviv et al. (2009) for comprehensive literature reviews on dynamic pricing and strategic customer behavior.

In the posted dynamic pricing scheme, a path of prices is announced to the customers at the beginning of the planning horizon. Since the posted pricing scheme eliminates the uncertainty of the

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future prices, it simplifies the decision making of strategic customers and, therefore, offers a simpler theoretical ground for more sophisticated dynamic pricing schemes. This pricing scheme is studied by a number of researchers such as Elmaghraby et al. (2008), Aviv and Pazgal (2008) and Dasu and Tong (2010). They investigate the best response for each customer, find the Nash equilibrium of the customers' purchasing strategies, and based on this equilibrium, they have extended the analysis to the impact of this strategic behavior on the seller's decision and the expected revenue.

In this paper, we focus on the uniqueness issue of the buyers' equilibrium bidding behavior. We consider a simple setting where two strategic customers compete for one unit of item offered by a monopolist in two periods. The two customers are identical in a sense that their valuations are drawn from the same distribution. Similar to Dasu and Tong (2010), the customers arrive at the start of selling season. Prices are preannounced in the beginning, and the second-period price represents markdown. Even in this simple setting, we find some examples showing multiple equilibria. Furthermore, we show the uniqueness of the equilibrium when the buyers' valuations are uniformly distributed.

While the unique equilibrium behavior is often essential to meaningfully predict the outcomes of a game with the strategic customers, this issue has received limited attention in the literature. For example, Dasu and Tong (2010) do not acknowledge that the buyer's equilibrium bidding strategy may not be unique. Osadchiy and Vulcano (2010) and Correa et al. (2011) have investigated uniqueness in their extensions to the two-period posted pricing scheme of Aviv and Pazgal (2008). Osadchiy and Vulcano (2010) offer a sufficient condition for the uniqueness of an equilibrium, and Correa et al. (2011) show the existence of unique equilibrium in a more general setting with one item. These papers assume that the buyers arrive dynamically according to a Poisson process, which makes it easier to establish the uniqueness of the equilibrium since the buyers bid sequentially (one at a time) instead of biding simultaneously. In our paper, we consider a setting where all of the buyers present from the beginning of the planning horizon, and show that multiple equilibria can exist even if the firm offers only one unit of the item. The model by Elmaghraby et al. (2008) adopts an assumption that "valuations are drawn from nonoverlapping intervals", which sidesteps the difficulty associated with proving uniqueness by assuming that only one of the two buyers faces a nontrivial bidding decision.

2 Model Description

Our model is adapted from Elmaghraby et al. (2008) and Dasu and Tong (2010). Suppose that there exist $\overline{N}$ buyers, indexed by $i = 1, \ldots, \overline{N}$, and each buyers is interested in obtaining one unit of a particular product. Let V^i denote the random variable representing buyer i's valuation of the product, and we use v^i to denote its realization. We assume that $\{V^1, \ldots, V^{\overline{N}}\}$ are independent and identically distributed with the common distribution V, and we refer to its cumulative density function and probability density function as $G(\cdot)$ and $g(\cdot)$, respectively. Let K be the number of units for sale. The price changes from the first period to the second period, and let p_1 and p_2 denote the price per unit in the first period and the second period, respectively. Both p_1 and p_2 are exogenously given, and we assume $p_1 \geq p_2$.

We describe the sequence of events. The value of v^i is realized for each i, and any customer i with valuation $v^i \leq p_2$ leaves the system immediately. Then, the remaining buyers decides, independently and simultaneously, whether or not to place a bid in the first period, N. If the seller (monopolist) receives K or more bids, then the she randomly selects the K bidders to whom the unit will be sold at the price of p_1, and there is no more unit remaining for the second period. If the number of bids is less than K, then each buyer who bid will buy the product at p_1, and any remaining unit will be sold in the second period at p_2, when buyers who did not buy in the first period have an opportunity to bid.

This model offers a simple structure to analyze the buyer's problem in the second period. If he does not place a bid, then his payoff will be 0. If he places a bid, then there is some chance being able to buy a unit, in which case his payoff is $v - p_2$, where v is the buyer's valuation of the product. Thus, it is optimal that the buyer places his bid if and only if his valuation exceeds p_2. The first period problem involves a more delicate tradeoff between the price and the probability of obtaining the product. Below, we characterize the buyers' equilibrium bidding strategies and show that they follow a threshold policy.

Lemma 1. *Fix p_1 and p_2, where $p_1 \geq p_2$. Let $i \in \{1, \ldots, \overline{N}\}$.*

(a) The dominant strategy of bidder in the second period is to bid if and only if $v^i \geq p_2$.

(b) Any equilibrium of among the buyers can be characterized by $\tau^i \geq p^1$, for each $i \in \{1, \ldots, \overline{N}\}$

such that bidder i's strategy is given by

$$\begin{cases} \textit{do not bid} & \textit{if } v^i < p_2 \\ \textit{bid in period 2} & \textit{if } p_2 \leq v^i < \tau^i \\ \textit{bid in period 1} & \textit{if } v^i \geq \tau^i \ . \end{cases} \tag{1}$$

Proof. If the second period bids are accepted, then the buyer's expected profit from bidding in the second period is $(v^i - p_2)$ multiplied by the probability that the buyer will obtain a unit in the second period conditioned on bids being accepted in the second period. Since this depends only on the other bidders' strategies, not on v^i, the optimal decision for buyer i is to bid if and only if $v^i \geq p_2$. This proves (a).

Now, it suffices to consider the first period bid for the case $v^i \geq p_2$. Suppose that we fix the strategies of buyers other than buyer i. Let π_1 and π_2 denote the probability that buyer i will obtain a unit if he bids in the first period and if he bids in the second period, respectively. Clearly $\pi_1 \geq \pi_2$. Suppose that it is optimal to bid in the first period with v^i. Then,

$$\pi_1 \cdot (v^i - p_1) \ \geq \ \pi_2 \cdot (v^i - p_2).$$

Then, for any $\hat{v} > v^i$, we have

$$\begin{aligned} 0 &\leq \pi_1 \cdot (v^i - p_1) - \pi_2 \cdot (v^i - p_2) \\ &\leq \pi_1 \cdot (v^i - p_1) - \pi_2 \cdot (v^i - p_2) \ + \ (\pi_1 - \pi_2) \cdot (\hat{v} - v^i) \\ &= \pi_1 \cdot (\hat{v} - p_1) - \pi_2 \cdot (\hat{v} - p_2) \ , \end{aligned}$$

where the second inequality follows from $\pi_1 \geq \pi_2$ and $\hat{v} > v^i$. Thus, we obtain $\pi_1 \cdot (\hat{v} - p_1) \geq \pi_2 \cdot (\hat{v} - p_2)$, implying that it is also optimal for buyer 1 to bid in the first period when his value is $\hat{v}$. □

We remark that Lemma 1 is applicable for a general model with arbitrary $\overline{N}$, K and any distribution for V.

3 Multiplicity and Uniqueness of the Buyers' Equilibrum Strategy

3.1 Multiplicity of Equilibrium

We first show that the equilibrium strategy for buyers may not be unique.

Proposition 2. *Suppose $\overline{N} = 2$ and $K = 1$. Suppose that the buyers' valuation is deterministic and identical, i.e., $V^1 = V^2 = \hat{v}$. If $p_1 \leq \hat{v} \leq 2p_1 - p_2$, then the buyers' bidding strategy forms multiple equilibria.*

Proof. If both players bid in the first period, then two buyers have an equal probability of obtaining the object. In this case, the payoff to a buyer i is $V^i - p_1 = \hat{v} - p_1$ if the buyer obtains the object; otherwise, it is 0. Thus, the expected payoff is $(\hat{v} - p_1)/2$. If only one buyer submits the bid in the first period, this buyer will have the payoff of $\hat{v} - p_1$, whereas the other buyer has the payoff of 0. If neither buyers submits the bid in the first period, each buyer receives the object in the second period with probability 0.5, yielding the expected payoff of $(V^i - p_2)/2 = (\hat{v} - p_2)/2$. We summarize this in the following table.

(Payoff to 1, Payoff to 2)	Buyer 2 bids in period 1	Buyer 2 bids in period 2
Buyer 1 bids in period 1	$\left(\frac{\hat{v}-p_1}{2}, \frac{\hat{v}-p_1}{2}\right)$	$(\hat{v} - p_1, 0)$
Buyer 1 bids in period 2	$(0, \hat{v} - p_1)$	$\left(\frac{\hat{v}-p_2}{2}, \frac{\hat{v}-p_2}{2}\right)$

Note that there are two equilibria above since $p_1 \leq \hat{v} \leq 2p_1 - p_2$: (i) both buyers bid in the first period, and (ii) both buyers do not bid in the first period and bid in the second period. (This is essentially the well-known *matching pennies* game.) $\square$

We make three remarks on the multiplicity of equilibria. (i) The result of Proposition 3.1 can be extended to non-identical valuations between the buyers. (ii) Multiple equilibria can exist even when the valuations are uncertain. (iii) Under the setting of Proposition 2, it can be shown that an equilibrium exists in *mixed* strategies, where each of the buyers bid in the first period with probability $\frac{2p_1-p_2-\hat{v}}{p_1-p_2}$.

3.2 Existence of the Unique Equilibrium: Uniform Distribution Case

Now, we focus our attention to the case with $\overline{N} = 2$ and $K = 1$ where each V^i is uniformly distributed between 0 and 1, for $i \in \{1, 2\}$. In this section, we show that a unique equilibrium exists for buyers' strategies, and provide a close-form expression for the equilibrium bidding strategy.

Suppose $i, j \in \{1, 2\}$ where $i \neq j$. Based on Lemma 1(b), the threshold values τ^i and τ^j play an important role in describing the strategy, where $\tau^i, \tau^j \in [p_1, 2]$. Suppose we fix the value of τ^j, and study the optimal response of buyer i. Suppose $v^i \in [p^1, 1]$. The buyer needs to decide between bidding in the first period and bidding in the second period. Let $\tau^j \in [p_1, 1]$.

- Suppose buyer i bids in the first period. Then, if buyer j's valuation $v^j < \tau^j$, then buyer i obtain the unit. If $v^j \in [\tau^j, 1]$ then buyer i obtains the unit with probability 1/2. Thus, the expected profit is given by

$$u_1^i(\tau^j) \quad = \quad \left[\tau^j + (1 - \tau^j)/2\right] \left(v^i - p_1\right) \quad = \quad \left(\frac{\tau^j}{2} + \frac{1}{2}\right) \left(v^i - p_1\right) \ .$$

- Suppose buyer i does not bid in the first period, and bids in the second period. Then, buyer i obtains the unit only if buyer j did not bid in the first period. In this case, if buyer j does not bid in the second period (which occurs if $v^j < p_2$), then buyer i will get the unit. Otherwise, if buyer j bids in the second period (i.e., $v^j \in [p_2, \tau^j)$), then buyer i will get the unit with probability 1/2. Thus, the expected profit in this case is given by

$$u_2^i(\tau^j) \quad = \quad \left[p_2 + (\tau^j - p_2)/2\right] \left(v^i - p_2\right) \quad = \quad \left(\frac{\tau^j}{2} + \frac{p_2}{2}\right) \left(v^i - p_2\right) \ .$$

Then, the buyer i's decision is based on comparing $u_1^i(\tau^j)$ and $u_2^i(\tau^j)$. Based on this approach, we obtain the following result.

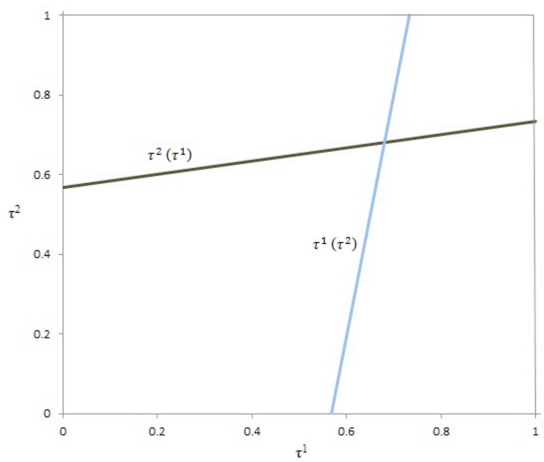
Proposition 3. *Suppose $\overline{N} = 2$, $K = 1$, and $V^i \sim U[0, 1]$ for $i \in \{1, 2\}$. Fix p_1 and p_2, where $p_2 < 1$. Then, the unique equilibrium bidding strategy is symmetric and characterized by Lemma 1(b) and the following:*

(i) if $\frac{p_1 - p_2^2}{1 - p_1} \in [p^1, 1]$, then $\tau^1 = \tau^2 = \frac{p_1 - p_2^2}{1 - p_1}$; and

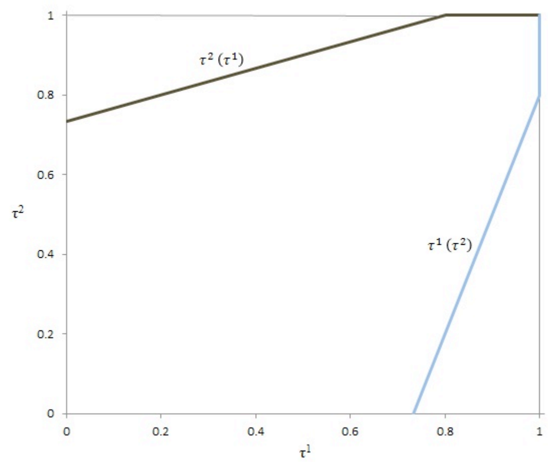
(ii) if $\frac{p_1 - p_2^2}{1-p_1} > 1$*, do not bid in the first period.*

Furthermore, $\frac{p_1 - p_2^2}{1-p_1} \geq p^1$.

The proof of Proposition 3 appears in the appendix. This proposition provides not only a closed-form expression for an equilibrium strategy, but also shows that this equilibrium is unique and symmetric. See Figure 1 for illustration.



(a) $(p_1, p_2) = (0.5, 0.4)$



(b) $(p_1, p_2) = (0.6, 0.4)$

Figure 1: Uniform Distribution Examples. In (a), the equilibrium $\tau_1 = \tau_2$ value is 0.68. In (b), the the equilibrium $\tau_1 = \tau_2$ value is 1, i.e., not bidding in the first period.

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A Appendix

A.1 Proof of Proposition 3

Proposition 4. *Under the conditions of Proposition 3, if bidder j's strategy follows by a threshold policy of* (1) *with fixed* $\tau^j \in [p_1, 1]$*, then bidder i's best response strategy also follows* (1) *where the threshold value* τ^i *satisfies*

$$\tau^i(\tau^j) \quad = \quad \frac{p_1 - p_2^2}{1 - p_2} + \frac{p_1 - p_2}{1 - p_2}\tau^j \ .$$

Proof. We compare $u_1^i(\tau^j)$ and $u_2^i(\tau^j)$. Both of these functions are linear functions of v^i with an intersection at $\tau^i(\tau^j)$. To ses this, we equate $u_1^i(\tau^j)$ and $u_2^i(\tau^j)$ and solve for v^i:

$$\begin{aligned}
\left(\frac{\tau^j}{2} + \frac{1}{2}\right)(v^i - p_1) &= \left(\frac{\tau^j}{2} + \frac{p_2}{2}\right)(v^i - p_2) \\
\left(\frac{1 - p_2}{2}\right)v^i &= \frac{p_1 - p_2^2}{2} + \left(\frac{p_1 - p_2}{2}\right)\tau^j \\
v^i &= \frac{p_1 - p_2^2}{1 - p_2} + \left(\frac{p_1 - p_2}{1 - p_2}\right)\tau^j \ .
\end{aligned}$$

Note that $u_1^i(\tau^j)$ has a higher slope than $u_2^i(\tau^j)$. Thus, if $v^i < \tau^i(\tau^j)$, then $u_1^i(\tau^j) > u_2^i(\tau^j)$, i.e., it is better to bid in the first period; similarly, if $v^i > \tau^i(\tau^j)$, then $u_1^i(\tau^j) < u_2^i(\tau^j)$, i.e., it is better to bid in the second period. This completes the proof. $\square$

Proof of Proposition 3. We first note that

$$\frac{p_1 - p_2^2}{1 - p_1} \geq \frac{p_1 - p_1^2}{1 - p_1} = p_1 .$$

We now argue that (i) and (ii) form an equilibrium bidding strategy.

- Suppose $\frac{p_1 - p_2^2}{1-p_1} \in [p^1, 1]$. If $\tau_j = \frac{p_1 - p_2^2}{1-p_1}$, then from Proposition 4, buyer i's best response satisfies

$$\begin{aligned}
\tau^i(\tau^j) &= \frac{p_1 - p_2^2}{1 - p_2} + \frac{p_1 - p_2}{1 - p_2}\tau^j = \frac{p_1 - p_2^2}{1 - p_2} + \frac{p_1 - p_2}{1 - p_2} \cdot \frac{p_1 - p_2^2}{1 - p_1} \\
&= \frac{1 - p_1}{1 - p_1} \cdot \frac{p_1 - p_2^2}{1 - p_2} + \frac{p_1 - p_2}{1 - p_2} \cdot \frac{p_1 - p_2^2}{1 - p_1} = \frac{(1 - p_2)(p_1 - p_2^2)}{(1 - p_1)(1 - p_2)} \\
&= \frac{p_1 - p_2^2}{1 - p_1} = \tau^i .
\end{aligned}$$

  Thus, (τ^1, τ^2) forms an equilibrium strategy.

- Now, suppose $\frac{p_1 - p_2^2}{1-p_1} > 1$. If buyer j's strategy is not to bid in the first period, and it is equivalent to having $\tau_j = 1$. Thus, bidder i's best response satisfies

$$\tau^i(\tau^j) = \frac{p_1 - p_2^2}{1 - p_2} + \frac{p_1 - p_2}{1 - p_2}\tau^j = \frac{p_1 - p_2^2}{1 - p_2} + \frac{p_1 - p_2}{1 - p_2} > \frac{1 - p_2}{1 - p_2} = 1 ,$$

  where the above inequality follows from

$$(p_1 - p_2^2) + (p_1 - p_2) > (1 - p_1) + (p_1 - p_2) = 1 - p_2 .$$

  Thus, bidder i's strategy is not to bid in the first period.

We now argue that any equilibrium bidding strategy should be symmetric. Let $(\hat{\tau}^1, \hat{\tau}^2)$ be any equilibrium bidding strategy, where $\hat{\tau}^1, \hat{\tau}^2 \in [p_1, 1]$. Assume, by way of contradiction, $\hat{\tau}^1 \neq \hat{\tau}^2$. Without loss of generality, assume $\hat{\tau}^1 < \hat{\tau}^2$; thus $\hat{\tau}^1 < 1$.

Then, from Proposition 4, we must have

$$\hat{\tau}^1 \; < \; \tau^2(\hat{\tau}^1) \;\; = \;\; \frac{p_1 - p_2^2}{1-p_2} + \frac{p_1 - p_2}{1-p_2}\hat{\tau}^1 \; .$$

(Otherwise, we would not have $\hat{\tau}^1 < \hat{\tau}^2$.) Then,

$$\tau^1(\hat{\tau}^2) \;\; = \;\; \frac{p_1 - p_2^2}{1-p_2} + \frac{p_1 - p_2}{1-p_2}\hat{\tau}^2 \; > \; \frac{p_1 - p_2^2}{1-p_2} + \frac{p_1 - p_2}{1-p_2}\hat{\tau}^1 \; > \; \hat{\tau}^1 \; .$$

This implies that $\hat{\tau}^1$ is not the best response strategy to $\hat{\tau}^2$. This contradiction shows $\hat{\tau}^1 = \hat{\tau}^2$.

Finally, we argue for the uniqueness of the equilibrium. Suppose $(\tilde{\tau}, \tilde{\tau})$ and $(\hat{\tau}, \hat{\tau})$ represent two distinct equilibrium strategies, where $\tilde{\tau}, \hat{\tau} \in [p_1, 1]$. Suppose, by way of contradiction, that $\tilde{\tau}^1 \neq \hat{\tau}^1$. Without loss of generality, there exists $\delta > 0$ such that $\hat{\tau} \; = \; \tilde{\tau} + \delta$. Note that this implies $p_1 < 1$. From Proposition 4,

$$\tau^i(\tilde{\tau}) \; = \; \frac{p_1 - p_2^2}{1-p_2} + \frac{p_1 - p_2}{1-p_2}\tilde{\tau}^j \qquad \text{and} \qquad \tau^i(\hat{\tau}) \; = \; \frac{p_1 - p_2^2}{1-p_2} + \frac{p_1 - p_2}{1-p_2}\hat{\tau}^j \; .$$

- Case $\hat{\tau} < 1$. Then, $\tau^i(\tilde{\tau}) = \tilde{\tau}$ and $\tau^i(\hat{\tau}) = \hat{\tau}$. It follows

$$\delta \;\; = \;\; \tau^i(\hat{\tau}) - \tau^i(\tilde{\tau}) \;\; = \;\; \frac{p_1 - p_2}{1-p_2}\left[\hat{\tau} - \tilde{\tau}\right] \;\; = \;\; \frac{p_1 - p_2}{1-p_2}\delta \; < \; \delta \; .$$

- Case $\hat{\tau} = 1$. Then, $\tau^i(\tilde{\tau}) = \tilde{\tau}$ and $\tau^i(\hat{\tau}) \geq 1 = \hat{\tau}$. We have

$$\delta \;\; = \;\; \hat{\tau} - \tilde{\tau} \; \leq \; \tau^i(\hat{\tau}) - \tau^i(\tilde{\tau}) \;\; = \;\; \frac{p_1 - p_2}{1-p_2}\left[\hat{\tau} - \tilde{\tau}\right] \;\; = \;\; \frac{p_1 - p_2}{1-p_2}\delta \; < \; \delta.$$

In both cases, we obtain a required contradiction. Thus, we conclude that $\tilde{\tau} = \tilde{\tau}$. □