Dynamic pricing when consumers are strategic: Analysis of a posted pricing scheme

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We study dynamic pricing policies for a monopolist selling perishable products over a finite time horizon to buyers who are strategic. Buyers are strategic in the sense that they anticipate the firm's pricing policies. We are interested in situations in which auctions are not feasible and in which it is costly to change prices. We begin by showing that unless strategic buyers expect shortages dynamic pricing will not increase revenues. We investigate two pricing schemes that we call posted and contingent pricing. In the posted pricing scheme at the beginning of the horizon the firm announces a set of prices. In the contingent pricing scheme price evolution depends upon demand realization. Our focus is on the posted pricing scheme because of its ease of implementation. In equilibrium, buyers will employ a threshold policy in both pricing regimes i.e., they will buy only if their private valuations are above a particular threshold. We show that a multi-unit auction with a reservation price provides an upper bound for the expected revenues for both pricing schemes. Numerical examples suggest that a posted pricing scheme with two or three price changes is enough to achieve revenues that are close to the upper bound. Counter to intuition we find that neither a posted pricing scheme nor a contingent pricing scheme is dominant. The difference in expected revenues of these two schemes is small. We also investigate whether or not it is optimal for the seller to conceal inventory and sales information from buyers. A firm benefits if it reveals the number of units it has for sale and subsequently withholds information about the number of units sold at different prices.

Subject classifications: dynamic pricing; customer strategic behavior.
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1. Introduction.

In this paper we study a monopolist selling a fixed quantity of a perishable product over a finite time horizon. Examples of such products include airline tickets, fashion goods, and tee times on golf courses. Two important feature common to these products are (i) that replenishment may not be possible and that even if it is, costs may be too high, and (ii) that any unsold product at the end of the horizon is either worthless or of considerably lower value.

One strategy for increasing revenues is to segment the market and to offer the product at a different price to each segment. In the airline industry, the market can be segmented on the basis of purchase time – willingness to pay for a flight ticket increases as you get closer to departure time. When the seller cannot effectively segment the market, an alternative approach for increasing revenue is to dynamically change prices. Of course, if segmentation is possible, dynamic pricing can also be used in conjunction with segmentation strategies. Dynamic pricing is inter-temporal price discrimination, with a focus on demand uncertainty.

Work that has preceded ours in the sub-field of dynamic pricing1. , implicitly assumes that customers do not anticipate prices and behave myopically (Bitran and Mondschein (1997)). We argue that many customers are aware of pricing paths and that they time their purchases. Evidence of customers' strategic behavior abound; customers waiting for after-Christmas sales, anticipating price mark-downs of fashion goods and electronic products (McWilliams (2004)), and tracking

prices of airline tickets, are just a few examples. Lazear (1986) suggests that consumers can be divided into (i) shoppers who are exploring prices, and (ii) buyers who are ready to purchase.

We try to shed light on the influence of customers' strategic behavior on seller's equilibrium pricing policies. Auction theory, and more specifically literature on mechanism design is directly concerned with this question (Krishna (2002)). There are, however, many retail settings in which mechanisms such as auctions are not practical. Changing prices in retail outlets is a complex process that consumes resources (Levy et al. (1997)) and brick and mortar retailer are not set up to handle bids from buyers. Therefore, we are interested in mechanisms that involve a small number of price changes. We consider a number of pricing strategies. The simplest policy would be to announce and commit to a set of prices at the beginning of the selling horizon 2. . At the other extreme, the firm can change prices after observing sales. In addition, it can either reveal its sales and inventory levels to all buyers or conceal this information. In this paper, we study two different pricing policies we call posted and contingent. In the posted pricing policy, the firm commits to a price path at the beginning of the season. In the contingent pricing policy, the firm determines its price based on realized sales. In both schemes, the firm commits to the number of price changes3. .

We assume that all buyers are present at the beginning of the selling horizon and ignore uncertainty in the arrival process. All buyers are strategic and not just a subset as suggested by Lazear (1986), and they constantly monitoring prices. This is the polar extreme of the assumption traditionally made in the revenue management literature that none of the buyers are strategic. We believe that examining this extreme is valuable to isolate the impact of strategic behavior.

We assume that a single product is being sold and consumers differ only in the value they place on the product. The buyers and sellers have information about the number of units for sale, the size of the market, the number of price changes, and the distribution of valuations. Our analysis readily incorporates uncertainty in the size of the market. We assume that the duration of the selling horizon is short and ignore discounting. Because all buyers are present at the beginning, the actual duration of the selling period is inconsequential and all that matters is the number of price changes.

Presence of strategic buyers raises a number of interesting questions. What are the consequences on revenues if a firm ignores strategic behavior? What is the impact of strategic behavior on pricing policies? What is the loss in optimality if the firm commits to posted prices? How many price changes are needed? How do factors such as the size of the market, level of supply, uncertainty in the size of the market, and the distribution of the valuations influence the performance of the different pricing policies? How many units should the firm stock? Should the firm reveal the number of units it has for sale? Should the firm conceal information about the number of units sold at different prices?

1.1. Main Findings

The characteristics of optimal dynamic pricing policies and the ability of the firm to extract consumer surplus are significantly altered when consumers anticipate pricing policies. For one thing, it is costly for the firm to ignore such behavior when developing pricing policies. Pricing schemes that account for strategic buyers are qualitatively different. The gap between the lowest price and the highest price is reduced. If consumers are not strategic, it is well known (Gallego and van Ryzin (1994)) that dynamic pricing will increase a firm's revenues if there is uncertainty in the customer arrival process. On the other hand, if buyers are strategic then even if there is uncertainty in demand, dynamic pricing need not increase the firm's revenues. For dynamic pricing to be useful it is essential that consumers anticipate a shortage. Static pricing is optimal regardless of whether or not demand is uncertain, as long as buyers are assured of supply. Further, when buyers are strategic and shortages are perceived, dynamic pricing is better than static pricing even if demand is deterministic.

A strategic buyer, except in the terminal period, will buy the product only if his or her valuation is above a threshold. This threshold, except at the lowest price, is strictly greater than the prevailing price. When the valuation is close to the prevailing price, a buyer may find it worth while to wait for a lower price, even though there is a risk of a stock-out. Models that ignore strategic behavior assume that everyone whose valuation is above the given price will buy the product. The threshold depends on the size of the market, and the level of scarcity. We use the ratio of the number of units for sale(K) to the number of customers (N) as a measure of the level of scarcity.

In the limit as the number of price changes approaches infinity, posted and contingent pricing schemes maximize expected revenues. They are revenue-equivalent to a multi-unit auctions with reservation prices. This linkage enables us to derive an upper bound for expected revenues. When the number of prices changes is restricted the loss in optimality depends on the level of scarcity $(\frac{K}{N})$, the distribution of valuations, and the size of the market. If $\frac{K}{N} \geq 1$ or $\frac{K}{N}$ is close to zero, a single price is optimal or nearly optimal, respectively. Also as $N \to \infty$, regardless of the ratio $\frac{K}{N}$, no more than one price change is needed to maximize expected revenues.

When prices can be changed limitlessly, expected revenues are not effected by whether or not buyers are aware of the number of units remaining unsold. However, when the number of price changes is finite information about the sales and inventories influence expected revenues. Based on our numerical examples we conjecture that the firm should reveal the number of units that are available for sale at the beginning of the season, but subsequently conceal the inventory levels.

Contingent pricing clearly dominates posted pricing when consumers are not strategic. This is no longer true when buyers anticipate prices. Neither scheme is dominant. The difference in expected revenues, however, is quite small. Overall our numerical examples suggest that simple policies of pre-announcing a few prices in advance achieves near optimal revenues. This is particularly true when there are more than 30 buyers or the number of units for sale is in excess of 50% of the market size.

1.2. Organization of the Paper

In the next section, we review related literature. In section 3, we introduce our model and show that customers have to perceive a shortage for dynamic pricing to be useful. In section 4, we analyze buyers' equilibrium strategies and show that in all of the pricing schemes there is a unique equilibrium. In section 5, we derive properties of the pricing policies and show that in the limit as the number of price changes approaches infinity both schemes are optimal and are equivalent to a descending price auction with a reservation price. We also derive sufficient conditions under which it is optimal for the firm to conceal sales information. Section 6 contains numerical examples that assess the value of multiple price changes, compares the two pricing schemes, and explores the benefits of concealing inventories. We conclude in section 7. All proofs are in the appendix.

2. Literature Review

Economists were among the earliest researchers to study pricing when customers behave strategically. This work was concerned with consumer durables, and ignored capacity and inventory constraints. The well-known Coase theorem (Coase (1972)) considers a monopolist selling a consumer durable. In equilibrium, a declining price path is subgame perfect, and rational customers anticipating the pricing path would not buy the product until the last period (Stockey (1981)). The price steadily declines to marginal cost, thereby eliminating all monopoly power. If customers tradeoff the value of consuming right away against the benefits of differing consumption for a lower price then customers with high valuations would buy in the early periods while low valuation customers would buy in a later period.

Researchers in marketing have also studied optimal pricing strategies when buyers anticipate prices. The price-skimming phenomenon has been investigated by Besanko and Winston (1990)).

In their model, customers' valuations are assumed to be uniformly distributed. The authors characterize a subgame perfect Nash equilibrium involving a rational seller and rational consumers and establish the optimality of a declining price path. Through numerical examples, they also demonstrate that a seller who ignores consumer strategic behavior will observe a substantial decline in profits. Unlike our work, papers in marketing are not concerned with perishable assets (Besanko and Winston (1990), Moorthy (1988)). They assume that the manufacturer can produce additional product in each period.

The dynamic pricing and revenue management literature is concerned with pricing perishable assets that cannot be replenished. This literature is vast, and a recent book by Talluri and van Ryzin (2004b) provides an excellent overview of this literature. Readers are also referred to reviews by McGill and van Ryzin (1999)), Bitran and Caldentey (2003), and Elmaghraby and Keskinocak (2003)). Early work in dynamic pricing did not consider strategic behavior. In recent years many papers have been developed on dynamic pricing that take into account consumer behavior. Several papers consider how consumers select among different products (Talluri and van Ryzin (2004a), Zhang and Cooper (2005), Savin and Xiao (2006),and Maglaras and Meissner (2006)). These papers are concerned with selection among alternative products, but do not worry about consumers timing their purchases.

Arnold and Lippman (2001) study a posted pricing problem. The firm announces a price and spends money to stimulate demand. A random number of customers arrive, and all buyers whose valuations exceed the prevailing price buy the product. They show that a declining price path is optimal. The key difference between our work and theirs is that in their model consumers do not anticipate price changes. We argue that the number of price changes are finite because it is expensive for the firm to change prices. Arnold and Lippman (2001) explicitly consider this cost and we do not. Nevertheless, in our model the seller can determine the expected revenues for different number of price changes and then select the optimum number.

Su (2006)) considers strategic buyers who time their purchases. In his model buyers arrive at a constant deterministic rate. Buyers fall into four groups depending on their valuation (high or low) and patience level (high or low). The number of buyers in each of the four groups is known for certain. He shows that whether or not prices increase or decrease depends on how the total population is distributed among these four groups. A declining price path is optimal if there are many impatient high value customers or there are many patient low value customers. The converse is true in the other cases. In our model pricing decisions by the seller and the purchase decisions by the buyer are driven by the uncertainty in demand. Su (2006)) model is a deterministic model. Due to these difference, the insights obtained by him are also very different from those in this paper.

Liu and van Ryzin (2006) also study a deterministic demand model that incorporates customers' strategic behavior. They are concerned with rationing policies. Buyers are risk averse and decide when to buy a single unit of the product. The prices are exogenous to their model, and are known to everyone. Thus they also study a posted pricing scheme. They explore whether or not a firm should ration the amount of stock available for sale at different prices. We derive optimal prices under uncertainty and do not address whether or not a firm should ration. We, however, show that revenues are maximized under both pricing schemes we study here, if we permit the number of price changes are allowed to grow to infinity.

Dynamic pricing when consumers are strategic is akin to auctions with a restriction on the number of price changes. We therefore draw from the auction literature (see Milgrom (2004)). Harris (1981) was the first to identify mechanisms that maximize seller's earnings. He showed that when the number of buyers exceeds the number of items available for sale a Vickery auction with an appropriate reserve price is optimal. Employing the revenue equivalence principle we show quite easily that if prices can be changed an infinite number of times then in the limit both of the dynamic pricing schemes we consider here maximize seller's expected revenues.

3. Model Formulation

We have a monopolist selling K perishable objects to N risk-neutral consumers over a finite time horizon T. Each customer wants to buy at most one unit of the product. As stated earlier, in order to highlight the effect of consumers' surplus-maximizing behavior, we assume the following: (1) all customers arrive at the start of the selling season; (2) consumption takes place immediately upon purchase and there is no discounting; (3) since the initial capacity is assumed to be exogenous, the cost of the product is zero; and (4) consumers' values are independent identically distributed and are private. These values remain constant throughout the horizon.

The seller does not know each consumer's valuation, but she and all other buyers know the distribution of the valuations. Without loss of generality, we normalize the support of the valuations to the range [0,1] and denote the valuation distribution by $G(v)$, for $v \in [0,1]$. The seller and all of the buyers at the start of the selling horizon are aware of the number of units for sale (K) and the size of the market (N), the number of price changes (T), and the distribution of private valuations. For ease of exposition, we develop our analysis with the assumption that the number of buyers is known. Our analysis extends to situations in which N is a random variable, provided everyone has the same prior distribution. Because all buyers are present at the beginning and there is no discounting, the length of the selling horizon is irrelevant. With a bit of abuse of notation, we will use the term periods to denote price changes.

We begin by identifying conditions under which dynamic pricing is preferred to static pricing. We first study the role of capacity. If supply exceeds market size($K \geq N$) in a posted pricing scheme, it is easy to show that the optimal price should be a single price. In the contingent pricing scheme things appear a little more complex because buyers see an uncertain price path, and the "lowest" price may not be obvious. However, adopting the concept of rational expectation equilibrium first proposed by Stockey (1979), we show below that a single price is still optimal.

In a two-period model, the seller's pricing decision is a vector (P_1, P_2), in which P_i is the price in period i, with $i = 1, 2$ (note that period 2 follows period 1). In the contingent pricing scheme, the seller's pricing scheme should be denoted as $(P_1, f(P_2))$, where $f(.)$ is the probability density function of the period 2 price and would depend on P_1. In rational expectation equilibrium, buyers and the seller share the same knowledge about the distribution of the period 2 price.

LEMMA 1. *If $K \geq N$, for any given price P_1 , and if a consumer with valuation v^b finds it optimal to buy in period 1, then all consumers with valuation $v \geq v^b$ will also buy in period 1; if a consumer with valuation v^b decides to postpone purchase to period 2, all consumers with valuation $v < v^b$ will also postpone purchases to period 2 even if their valuation exceeds the current price.*

The following proposition establishes the fact that no matter what pricing scheme is adopted, a single price is optimal when supply exceeds demand.

PROPOSITION 1. *Let π_r be the optimal revenue for the seller under the pricing scheme $(P_1, f(P_2))$, and π_s be the optimal revenue for the seller in the single price scheme. If $K \geq N$ then $E(\pi_r) \leq E(\pi_s)$.*

We see that dynamic pricing is not useful when there is no shortage. The consumers' ability to time their purchases eliminates the possible benefits of inter-temporal price discrimination by the seller. Therefore, in the remainder of this paper, we restrict ourselves to the case in which supply is limited ($K < N$). Because the number of price changes is finite, it is conceivable that at some price the number of offers received by the seller exceeds the number of units for sale. Since the buyer cannot rank the buyers on their valuations, we assume that a proportional rationing mechanism is employed to allocate the objects.

We next study the role of demand uncertainty. Uncertainty in demand can be due to uncertainty in valuations and uncertainty in the number of buyers. Here we assume the number of buyers

is known. We investigate whether dynamic pricing is better than static pricing when demand is deterministic. The following example provides some insights.

EXAMPLE 1. Suppose a seller has 2 units of product to sell, and there are 10 buyers. The valuation vector is: (100,40,35,30,28,26,25,23,21,20). The optimal single price for the seller is 100, with a realized sale of 1 and a total revenue of 100. The clearing price of 40 yields a revenue of $40 \times 2 = 80$. Alternatively, the seller can pre-announce prices (82,20). The consumer with valuation 100 can get a surplus of 100-82=18 for sure in period 1, or can get an expected surplus of $(100 - 20) \times 2 \div 10 = 16$ in period 2, provided a proportional rationing mechanism is used. Obviously, the first unit will be sold at price 82 to the customer with valuation 100. As a result, the total revenue of the seller will be 82+20, which is larger than 100. We conclude that for this particular case a dynamic price path (82,20) is superior to the optimal single-price policy.

This example at least shows that a single price policy may not be optimal. However we may ask, since a single-pricing policy is not optimal in this deterministic demand case, what is the structure of an optimal pricing policy? How many price changes are needed? We answer this question in the following lemma, which is interesting in its own right.

LEMMA 2. *If demand is deterministic and demand exceeds supply, the optimal pricing scheme has no more than one price change.*

When demand is deterministic everyone is aware of the clearance price. At any price above the clearance price all bidders are assured of supply and at any price below the clearance price, the firm can liquidate all inventories. Thus in any optimal pricing scheme there can be at most one price above the clearance price and one price below the clearance price.

Based on these findings, in the remainder of this paper we assume that $K < N$ and that demand is uncertain.

4. Equilibrium Buying Behavior

4.1. Posted Pricing Scheme

In the posted pricing scheme, the seller announces and commits to a price path $(P_1, P_2, ..P_T)$. Clearly $P_1 \geq P_2 \geq ..P_{T-1} \geq P_T$. Let us begin with a two-period problem. We show next that the equilibrium strategy is for all buyers with valuations above a threshold (y) to purchase in the first period. In our analysis we treat the Nth buyer as the focal buyer. We derive the optimal response for this buyer given the strategies of the other $N - 1$ buyers.

Let:

i, j : Number of bidders in period 1 and period 2, respectively;

$Pr_1(i)$: Probability that $i - out - of - (N - 1)$ buyers bid in period 1;

$Pr_2(j|i)$: Probability that $j - out - of - (N - 1 - i)$ buyers bid in period 2 given that i buyers bid in period 1;

$\pi_1(y)$: Probability of the Nth buyer getting the product in period 1; and

$\pi_2(y)$: Probability of the Nth buyer getting the product in period 2.

PROPOSITION 2. *For any set of prices (P_1, P_2), the following characterizes the unique Bayesian Nash equilibrium:*

Let y^* be the smallest solution for equation(1) in the range $[P_1, 1]$. All buyers with valuations $v \in [y^*, 1]$ will bid in period 1, and buyers with valuation $v \in [P_2, y^*)$ will bid in period 2.

If equation(1) has no solution within the range $[P_1, 1]$, no one will bid for the product in period 1 and those with valuations in the range $[P_2, 1]$ will bid in period 2.

$$\pi_1(y)(y - P_1) = \pi_2(y)(y - P_2), \tag{1}$$

N	K	Pricing if Buyers Are Assumed to Be Myopic	Pricing if Buyers Are Assumed to Be Strategic
10	2	0.85,0.64	0.76,0.64
20	4	0.86,0.68	0.78,0.68
30	6	0.86,0.70	0.80,0.71
40	8	0.86,0.71	0.78,0.68
50	10	0.87,0.72	0.77,0.65

Table 1 Comparison of Optimal Prices, Buyers Distribution are U(0,1)

where:

$$\pi_1(y) = \sum_{i=0}^{k-1} Pr_1(i) + \sum_{i=k}^{N-1} \frac{k}{i+1} Pr_1(i)$$

$$\pi_2(y) = E_{I,J}[min(\frac{k-i}{j+1}, 1)] = \sum_{i=0}^{k-1} Pr_1(i) \sum_{j=0}^{j=N-1-i} Pr_2(j|i) \min(\frac{K-i}{j+1}, 1)$$

$$Pr_1(i) = \binom{N-1}{i}(1 - G(y))^i G(y)^{(N-1-i)} \quad for\ i = 0, 1, \ldots N-1$$

$$Pr_2(j|i) = \binom{N-1-i}{j}(1 - \frac{P_2}{G(y)})^j(\frac{P_2}{G(y)})^{(N-1-i-j)}\ for\ i = 0, ..K-1;\ j = 0, ..(N-1-i)$$

$$Pr_2(j|i) = 0,\ for\ i = K, ...N-1;\ j = 0, 1..(N-1-i)$$

(2)

A buyer with valuation v will expect a surplus of $(v - P_1)\pi_1(y)$ if he decides to purchase in period 1 and a surplus of $(v - P_2)\pi_2(y)$ if he decides to wait till period 2. In both periods, the probability of obtaining a unit of product depends upon the initial capacity level K, total population N, as well as the decisions of all other consumers. Buyers with valuations sufficiently above P_1 prefer to buy in period 1 to reduce the likelihood of not getting the product. If the valuation is close to the prevailing price, a buyer may find it worth while to wait for a lower price. The drop in price may compensate for the decrease in probability of successful getting the product.

The fact that only buyers whose valuations are strictly greater than P_1 are willing to purchase at the higher price is one of the consequences of strategic behavior. This clearly decreases the demand at the higher price and the expected revenues. The difference between the threshold level y and P_1 depends on the difference between P_1 and P_2. For a given P_2 as we increase P_1, y keeps increasing. If the gap between P_1 and P_2 increases beyond a critical point, no one will offer to buy at the higher price. Thus, a second major consequence of strategic behavior is a compression in the prices. At optimality, the gap between P_1 and P_2 will be smaller relative to what it would be if buyers were not strategic. Table 1 illustrates this point. In this example the valuations of all the buyers are uniformly distributed between 0 and 1.

The threshold policy derived above extends to a general T period problem.

PROPOSITION 3. *For a T-period pricing scheme $(P_1, \ldots, P_T)$, in each period t only buyers with valuations greater than or equal to a threshold $y_t^*(k_t)$ will bid for the product. The threshold depends upon the number of units (k_t) available for sale. For $1 \le t < T$, $y_t^*(k_t) \ge P_t$ and for $t = T$, $y_T^*(k_T) = P_T$.*

Proofs of these propositions hinge on the fact that, regardless of the strategy employed by other buyers, for any given buyer the expected probability of obtaining the product does not increase as the prices drop. This remains true even if N is a random variable. Hence a threshold policy remains an equilibrium strategy when N is a random variable.

PROPOSITION 4. *Let N be a random variable with probability density function $\eta(.)$. This distribution is common knowledge among all the buyers and the seller. For a T-period pricing scheme $(P_1, \ldots, P_T)$, in each period t only buyers with valuations greater than or equal to a threshold $y_t^*(k_t)$ will bid for the product. The threshold depends upon the number of units (k_t) available for sale. For $1 \leq t < T$, $y_t^*(k_t) \geq P_t$ and for $t = T$, $y_T^*(k_T) = P_T$.*

4.2. Contingent Pricing

In the contingent pricing scheme, the seller determines the price based on the inventory on hand and does not commit to a price path. The buyers anticipate prices based on realized sales. Despite the difference between contingent and posted pricing schemes, the structure of the equilibrium remains the same. For brevity we derive the equilibrium for a two-price setting. The analysis can be extended to multiple prices and uncertain N in a manner similar to that for the posted pricing scheme.

Let P_1 be the price in period 1, and P_{2i} be the optimal price to be charged in period 2 when $i - out - of - (N-1)$ other buyers bid in period 1; that is:

$$P_{2i} = \{p : \max_p p[\sum_{j=0}^{K-i} \binom{N-i}{j}(1 - F_1(p))^j F_1(p)^{(N-i-j)} j +$$
$$\sum_{j=K-i+1}^{N-i} \binom{N-i}{j})(1 - F_1(p))^j F_1(p)^{(N-i-j)}(K-i)]\} \tag{3}$$

Let β^i be the probability that the Nth buyer will get the product if $i - out - of - N - 1$ other buyers bid in period 1 and and the buyer decides to bid in period 2. Thus:

$$\beta^i = E_j[min(\frac{k-i}{j+1}, 1)] = \sum_{j=0}^{N-1-i} Pr_2(j|i)min(\frac{K-i}{j+1}, 1) \text{ for } i = 0, \ldots, K-1$$
$$\beta^i = 0 \text{ for } i = K, \ldots N-1$$

PROPOSITION 5. *For any prices P_1 the following characterizes the unique Bayesian Nash equilibrium. Let y^* be the smallest solution for equation (4) in the range $[P_1, 1]$. All buyers with valuations $v \in [y^*, 1]$ will bid in period 1. If equation(4) has no solution within the range $[P_1, 1]$, no one will bid for the product in period 1. In period 2 all those whose valuations exceed P_2 but are less than y^* will bid. P_2 is contingent upon sales at price P_1 and solves (3).*

$$\pi_1(y^*)(y - P_1) = \sum_{i=0}^{k-1} Pr_1(i)\beta^i(y - P_{2i}^*) \tag{4}$$

5. Analysis of the Optimal Pricing Schemes

One of our goals is to understand types of pricing policies that firms can employ when consumers anticipate prices. In particular, we want to evaluate the effectiveness of a posted pricing scheme that involves one or two price changes. More generally, we would like to shed light on the following questions:

1. What is the loss in revenues if a firm ignores strategic behavior?
2. Because scarcity drives dynamic pricing, what is the relationship between scarcity and the value of dynamic pricing?
3. What is the optimum stocking level when buyers are strategic?
4. How many price changes are needed?
5. How do the characteristics of the valuation distribution influence pricing decisions?

6. How effective is the posted pricing scheme as compared with the contingent pricing scheme?

7. How does uncertainty in the size of the market influence the performance of different pricing schemes?

8. What is the value of withholding inventory information from buyers?

Determining optimal posted prices entails solving non-linear optimization problems that involve polynomial equations of a high order. Unfortunately, this makes it difficult to gain analytic insights into many of these questions. As a result, we have to resort to numerical experiments, and we do so in the next section. Nevertheless, there are a handful of analytic insights that we are able to elicit and we begin with the asymptotic properties of these policies. We explore the structure of the policy when the number of price changes T approaches ∞ or the number of players in the market N approaches ∞, while holding the ratio K/N constant. As the number of price changes, T, approaches infinity, we can be sure that that allocations are made to those with the highest valuations. This is significant as it enables us to determine the expected revenues. Also, as N approaches infinity, the influence of uncertainty in valuations decreases, thereby simplifying the structure of the optimal policies. Asymptotic analysis, besides being of independent interest, provides us with an upper bound on expected revenues. Asymptotic analysis also gives us some insights into the optimal stocking level decisions. We are able to provide a bound on the maximum stock level for large markets (N). Finally, we derive sufficient conditions under which the firm is better off concealing inventory levels.

5.1. Asymptotic Analysis

As the number of price changes T approaches ∞, the posted and contingent pricing schemes resemble auctions. Accordingly, we draw on findings in auction theory to establish limiting properties. Based on work by Milgrom and Weber(2000), it is fairly straight-forward to show that a firm maximizes its revenues if it uses a first-price, simultaneous auction with a reserve price 4. . The posted and contingent pricing schemes also have the objective of maximizing a firm's revenues, and we are able to show that both of these schemes are revenue-equivalent to the optimal mechanism. Further, the lowest price charged in the limit in both schemes will be strictly greater than zero and will correspond to the reservation price in the first-price auction. These results are formally presented below, and for brevity we focus only on the posted pricing scheme. Let

$J(v_i) = [v_i - \frac{1-G_i(v_i)}{g_i(v_i)}]$,
V : vector of valuations of all N buyers
V_{-i}: vector of valuations of $N-1$ buyers other than i,
V_{-i}^j: j^{th} largest value in the vector V_{-i}, and
$y_i(V) = Max(J^{-1}(0), V_{-i}^K)$

PROPOSITION 6. *If distribution $G(.)$ is such that $J(.)$ is non-decreasing, then in a posted pricing scheme, as the number of price changes $T \to \infty$:*

1. *The lowest posted price $P_T \to v^*$, where $v^* = \{v : J(v) = 0, for\ v \in (0,1)\ \}$.*

2. *The buyers with K highest valuations will get the product, provided their valuations are greater than v^*.*

3. *The seller expected revenues are maximized*

Let $Q(v_i, V_{-i})$ denote the probability that the ith buyer gets the product when his valuations is v_i and the valuation of the others is V_{-i}. Set $Q(v_i, V_{-i}) = 1$ if $v_i > y_i(v_i, V_{-i})$ and 0 otherwise.

In the limit the seller's expected revenues are given by:

$$\sum_{i \in N} \mathbf{E}_V[J(v_i)Q_i(v_i, V_{-i})] \tag{5}$$

Note $\frac{1-G_i(v_i)}{g_i(v_i)}$ is the inverse of the hazard rate. If the valuation distribution $G(.)$ has a non-decreasing hazard rate then the virtual value function $J(.)$ is strictly increasing (Krishnan (2002)). An interesting consequence of strategic buyer behavior is that the lowest price charged will in general be greater than zero, even if we permit an unlimited number of price change. The lowest price charged will be such that the virtual valuation is zero, not the actual value. For example, if the valuations are uniformly distributed between zero and one, then the lowest price will 0.5. Proposition 6, also provides us with an upper bound on the expected revenues that is easy to compute. We will use this upper bound to numerically evaluate pricing schemes.

We have seen in section 3 that when demand is deterministic we need at most two prices. In the limit, as N becomes large we would intuitively expect, due to the law of large numbers, the effect of uncertainty to progressively diminish. Thus, it is useful to understand the impact of the size of the market on the pricing policies. We show in the following proposition that in the limit, as N approaches ∞ and K/N is held constant, we need at most two prices. This is the same result we had when there was no uncertainty in valuations.

PROPOSITION 7. *For any fixed ratio K/N, as N approaches ∞ the optimal pricing scheme involves at most two prices.*

In the next section, we use numerical experiments to provide an understanding of how rapidly a two-price scheme converges to the optimal value as the market size grows.

5.2. Optimal Stocking Levels

We have seen in section 3 that as the ratio $\frac{K}{N}$ approaches 0 or 1, a single price is adequate. In our numerical experiments we find that the performance of the pricing schemes depends upon this ratio. In this context, the optimal stocking ratios become relevant. Are the optimal stocking levels such that only a few price changes result in near optimal revenues? We can show that optimal revenues are concave increasing in the number of units K available for sale. We also find that as the number of buyers in the market N grows large the firm finds it optimal to stock no more than the fraction of the market that can afford the optimal reservation price.

PROPOSITION 8. *The optimal expected revenues are concave increasing in K for fixed N.*

PROPOSITION 9. *As N approaches ∞, the optimal stocking level approaches $N * (1 - G(v^*))$, where $v^* = \{v : [v_i - \frac{1-G_i(v_i)}{g_i(v_i)}] = 0, \text{for } v \in (0,1) \}$*

The optimal stocking level, in the limit, is a percentile of the valuation distribution. For example, for uniformly distributed valuations, the optimal stocking level is the 50th percentile. The inventory level in a newsvendor problem is also a percentile, but of the demand distribution. In the newsvendor problem, the uncertainty is in the number of buyers that will purchase at a given price. Here too there is uncertainty in demand at a given price, but we optimally change prices.

5.3. Value of Inventory Information

Should sellers reveal the number of units they have for sale? Should a seller reveal the number of units for sale and then conceal the actual sales information? If the seller chooses not to reveal the number of units available for sale then the firm forgoes the ability to price optimally based on available stock. Thus, hiding initial sales quantity may not benefit the seller. On the other hand in a posted pricing scheme, concealing the number of units sold at different prices impacts the seller and buyer in different ways. Under this scheme, the seller is setting prices based on anticipated sales. By concealing sales information, the seller is forcing the buyers to work with the same limited information set that the buyer used to determine prices. We conjecture that it always beneficial for the seller to conceal sales information, although we are unable to establish this analytically. We can only provide sufficient conditions under which hiding sales information benefits a seller.

The question of whether or not a firm should conceal sales data is equivalent to studying the impact on profits if prices are fixed but the number of units for sale is a random variable. For simplicity, and without loss of generality, let us suppose that inventory levels are believed to be either high or low. The probability of inventories being high will increase the propensity to wait for lower prices, and the possibility of the inventories being low will cause buyers to bid at higher prices. Since we are considering situations in which initial sales quantities (K), market size (N), and the distribution of the valuations $(G(.))$ are known, beliefs about the number of unsold units should be unbiased and well calibrated.

Below, we provide sufficient conditions under which concealing inventories increases expected revenues.

Let $\Pi(K, N, P_1, P_2, y)$ be expected profits for the firm when the threshold is y and the number units for sale is known to be K.

PROPOSITION 10. *Let us suppose that the buyers believe that with probability α the inventory level is K_1 and with probability $1 - \alpha$ the inventory level is K_2. With these assumptions and for posted prices P_1 and P_2, let the threshold value be given by $y(\alpha)$. If $y(\alpha)$ is convex and $\Pi(K, N, P_1, P_2, y)$ is concave in y, then a firm's expected profits are higher if it does not reveal inventory levels.*

The properties of two factors $y(.)$ and $\Pi(.)$ are central to proposition 10. Interestingly, $y(.)$ is related to buyers' behavior and $\Pi(.)$ depends upon the valuation distribution $G(.)$.

As α increases, the likelihood of the inventory being high increases. This should induce more buyers to defer their bid. Recall only buyers whose valuations exceed $y(\alpha)$ bid when the price is P_1. Consequently, as α increases, for a given set of prices the threshold level $y(\alpha)$ will increase. If $y(\alpha)$ is convex, then it means that the threshold levels will decrease rapidly as α drops below one; on the other hand, as α increases from zero the threshold level rises very slowly. Convexity of $y(.)$ implies that for most values of α the threshold level is "low". This, in turn, means that buyers bid more aggressively or act as if they are averse to stock-outs.

Proposition 10 only contains sufficient conditions for hiding sales information. We conjecture that for a posted pricing scheme the seller always benefits from concealing sales data. The above discussion also suggests that if buyers are risk averse this benefit is likely to be greater. In our numerical examples, we find that $y(.)$ is convex for valuations that are uniformly distributed. We do not assume risk aversion.

6. Numerical Experiments

We begin this section with examples that measure the cost of ignoring strategic behavior. Next, we investigate the loss in optimality due to limiting the number of price changes. We know from proposition 6 that as the number of price changes increases the performance of both the posted and contingent pricing schemes will improve, and in the limit attain optimality. This proposition also gives us the upper bound that will be used to determine the loss in optimality from limited price changes.

There are several factors that influence the performance of the pricing schemes. Two of them are market size and scarcity levels (K/N). We therefore vary these two parameters. The third factor that has a bearing on revenues is the distribution of customer valuations $(G(.))$. We consider three different valuation distributions: Uniform, Beta(8,2), and Beta(2,8). Beta (8,2) is skewed to the right, and under this distribution most of the buyers have high valuations. B(2,8) is skewed to the left. The fourth factor is uncertainty in the market size. Intuitively, we would expect the performance of a limited price change scheme to deteriorate if we introduce uncertainty in the number of buyers.

Finally, we present examples that explore whether or not the firm should conceal inventory levels. We have seen in the previous section that there are two different approaches to concealing
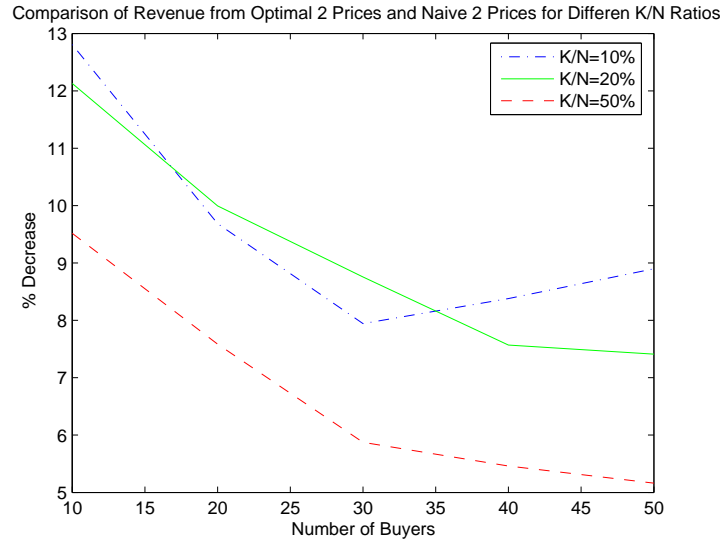
Figure 1   Drop in expected revenues when seller uses two posted prices without considering strategic behavior

inventory information. One method is for the firm to conceal the number of units for sale at the beginning of the selling horizon. An alternate approach is to truthfully reveal the number of units for sale at the beginning but withhold sales information. For uniformly distributed valuations, we study these two situations.

Figure 1 shows the consequences of ignoring strategic behavior. A firm that ignores strategic behavior will set prices based on the assumption that a buyer will purchase if his or her valuation exceeds the current price. We refer to prices set under this assumption as naive prices. Figure 1 shows the percentage drop in expected revenues if a seller employs naive prices instead of the optimal posted prices that consider strategic behavior. Here, valuations are assumed to be distributed uniformly between 0 and 1. For this distribution, the percentage loss in revenues varies between 5.5% and 13%. As the market size N increases, the percentage loss decreases. For higher K/N ratios, the loss in expected revenues also appears to be lower.

In Figure 2 to 6 we graph the percentage loss in optimality when a firm uses two or three posted prices. We compare the expected revenue from the posted price scheme with the maximum possible revenues (upper bound). For uniform distribution and three prices (Figure 3) the loss is no more than 3%. When the K/N ratio is close to 50% the percentage loss is generally under 1% for three prices. Figures 4 through 6 show the effect of the valuation distribution ($G(.)$). It is interesting to compare Figures 5 and 6. For Beta(8,2), which is right-skewed, we find that we are close to the upper bound with just two prices. On the other hand, when valuations are Beta(2,8) and the bulk of the buyers have lower valuations, the performance of the two-price scheme is considerably worse. In this case, it appears that you need more price changes. On the positive side, for all three distributions for N greater than 30, and with stocking levels of 50%, it appears that two posted prices are adequate.

The stocking decision will of course depend upon the product cost. Table 2 contains the optimal stocking levels for different costs, when $N = 20$ and the valuations are U(0,1).

We know from proposition 9 that for large N the stocking level depends upon the reservation price level v^*. When N is large even if the cost of the product is zero, the firm will not stock much more than $N * (1 - G(v^*))$, and the corresponding K/N ratio will be $1 - G(v^*)$. For $U(0,1)$, this limit is 50%, for Beta(8,2) (right skewed) the ratio is 13.5% and for Beta(2,8) (left skewed) the ratio is 92%. This suggests that when the distribution is left-skewed the stocking levels are going to be higher, decreasing the loss in optimality from a limited number of price changes.
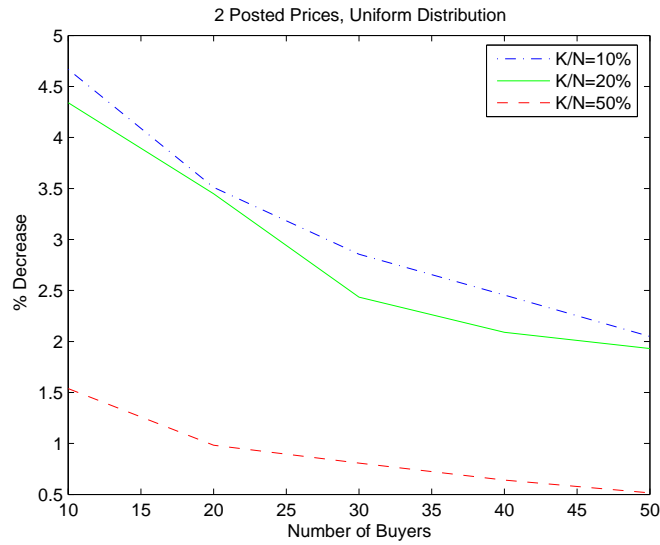
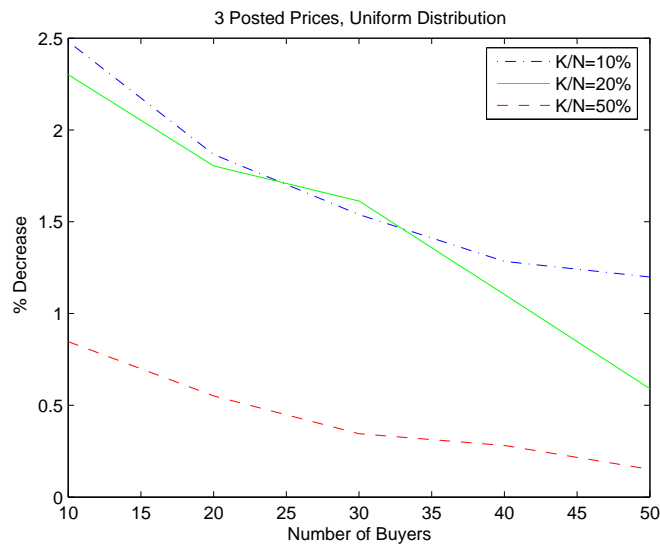**Figure 2**   % loss in optimality, 2 posted prices and uniform distribution

Figure 3   % loss in optimality, 3 posted prices and uniform distribution

Cost	Optimal K/N
0.1	50%
0.2	40%
0.3	35%
0.4	30%
0.5	25%

Table 2   N=20,Valuation U(0,1)

Let us now compare the posted pricing scheme with the contingent pricing scheme. If buyers are not strategic then clearly the contingent pricing scheme dominates the posted pricing scheme. However, this need not be true when buyers are strategic. A seller who adopts a posted pricing
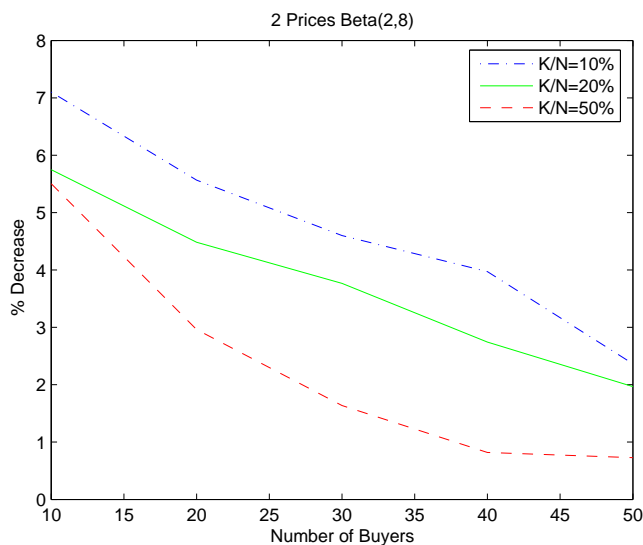
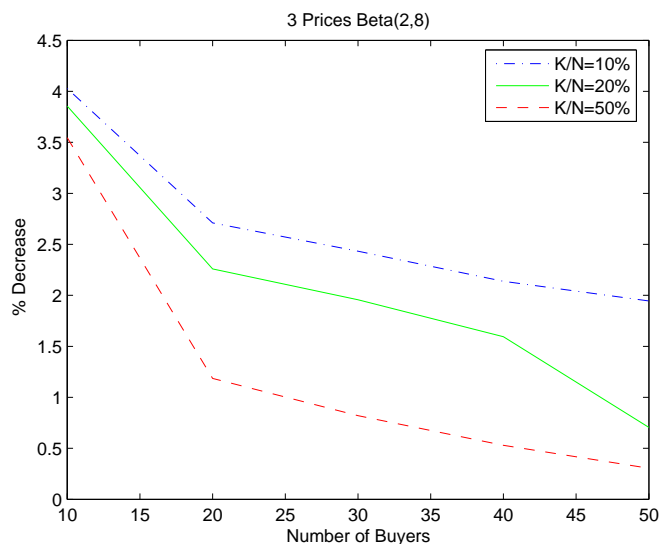**Figure 4**   % loss in optimality, 2 posted prices, Beta(2,8):left skewed

Figure 5   % loss in optimality, 3 posted prices, Beta(2,8):left skewed

forgoes some flexibility, but this commitment also eliminates options for the buyer. Thus, on balance, it is difficult to predict the impact on expected revenues. Our computational experiments suggest that neither procedure dominates. Figure 7 compares the percentage difference in expected profits between posted and contingent pricing schemes5.   for $N = 20$. Both schemes employ two prices. The differences between the two is small. The maximum difference we found is 1.6% and this difference will decrease further if either N or the number of price changes increases. This suggests that a posted pricing scheme may be adequate if the only uncertainty is that of buyer valuations.

For the seller, is there greater value to the option of delaying pricing decisions if there is uncertainty in the size of the market? Figure 8 sheds some light on this question. Here we assume that N is either small(7) or large (20) with probabilities α and $1 - \alpha$, respectively. The number of units for sale is fixed at 5. The two schemes are within one or two percentage points of each other, and
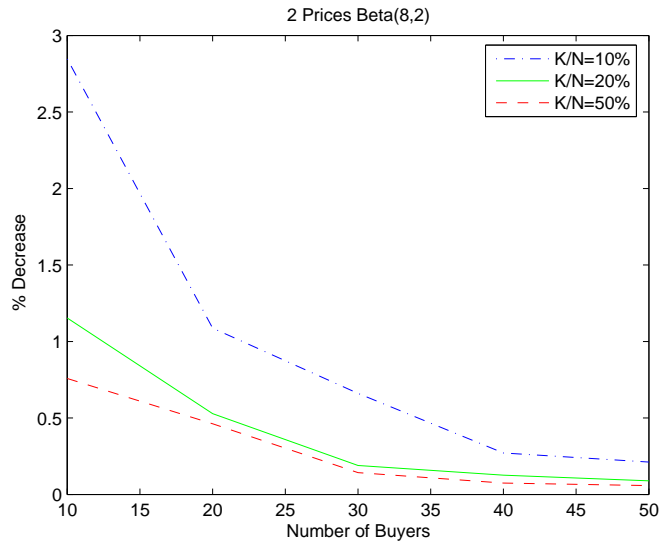
Figure 6   % loss in optimality, 2 posted prices Beta(8,2):right skewed
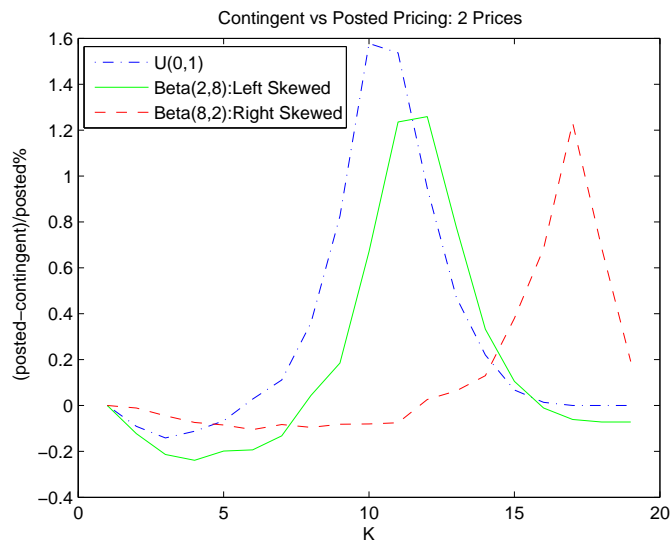

Figure 7   Comparison of Posted and Contingent Pricing Schemes, Fixed N

the relative performance seems to depend upon the expected value of the scarcity ratio (K/N). For higher levels of scarcity or lower values of K/N, contingent pricing appears to be marginally better than a posted pricing scheme. This little example does not, however, present a compelling reason to abandon a posted pricing scheme in favor of the contingent pricing.

The last two examples explore the role of inventory information. We begin with the question of whether or not a firm should conceal the number of units available for sale at the start of the season. We assume that the valuations are U(0,1) and that there are 20 buyers who all believe that the stock level is either 5 or 15 with probabilities α and $1 - \alpha$, respectively. These beliefs are unbiased and well calibrated. The firm, however, knows the actual stock level. In this situation the firm can either price based on the actual stock level or act as if it too does not know the true stock level but holds the same beliefs as the buyers. Once the prices are announced by the firm, buyers
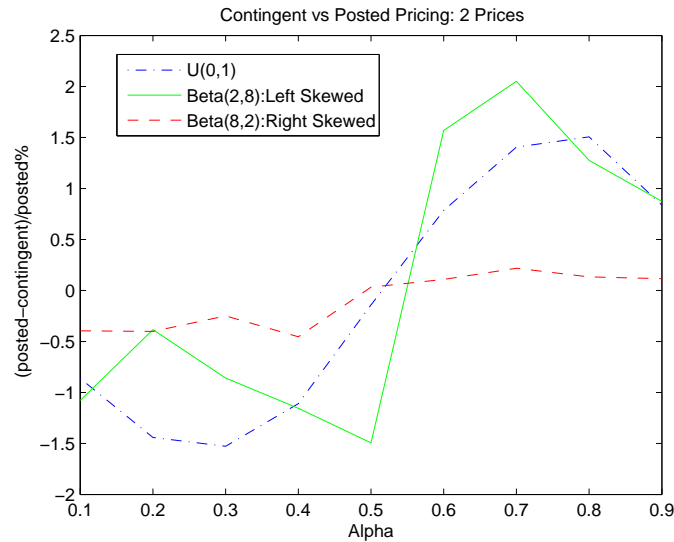
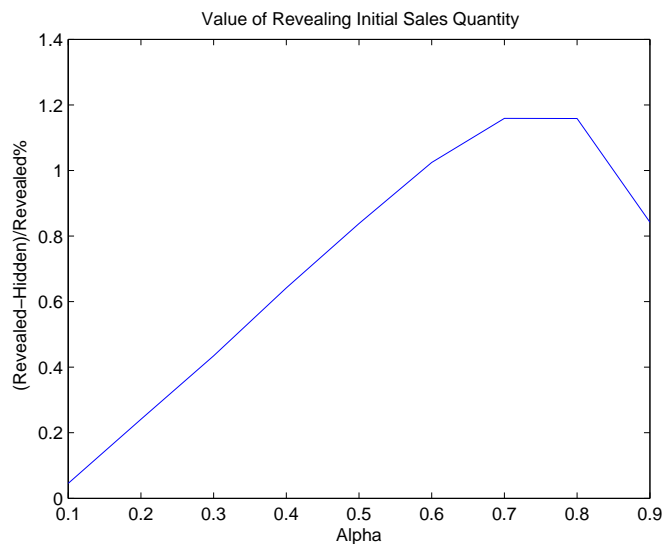**Figure 8**   Comparison of Posted and Contingent Pricing Scheme, Uncertain N

Figure 9   Value of Revealing Initial Sales Quantity, Valuations U(0,1), N=20

can discern the approach adopted by the firm. Figure 9 shows the change in expected revenues if the firm bases its prices on actual stock levels instead of treating stock levels as random variables. In this example, the firm is marginally better off if it reveals the stock levels.

In the last example we consider the question of whether the firm should hide sales information. Because this is an intermediary stage in the selling process, and because we are assuming posted prices, the firm can not change price based on the inventory level. We therefore fix prices and compute expected profits for different estimates of the number of units available for sale. Here $N = 12$ and $K = 2$ with probability α and $N = 12$ and $K = 4$ with probability $1 - \alpha$. The valuations are assumed to be U(0,1). In this case (Figure 10) we find that it is beneficial for the firm to hide sales information. Overall the value of inventory information does not seem significant.
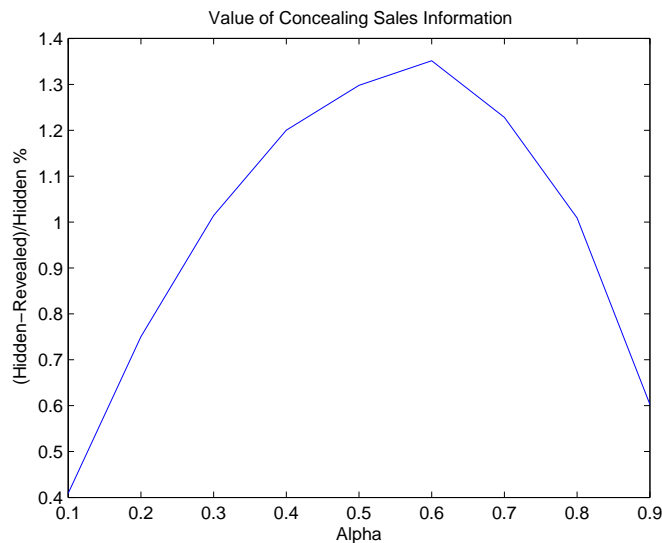
Figure 10   Value of Concealing Sales, Valuations Are U(0,1), N=20

7. Conclusion

The objective of this paper was to understand optimal pricing policies when consumers are strategic and it is difficult to change prices frequently. One of our main findings is that when buyers are strategic dynamic pricing is valuable only if buyers consider stock-outs as a possibility. Strategic buyers purchase a product only if their valuations exceed a threshold. This threshold is higher than the prevailing price and depends on the perceived level of scarcity. Thus at any given price fewer buyers will buy than a firm which ignores such behavior would expect to sell. More over, when buyers are strategic the the gap between the highest and the lowest prices is compressed. Stated another way, if a seller ignores strategic behavior while setting prices his or her initial price will be too high, and most buyers will prefer to wait for a price drop. We also find that, given any level of supply, as the size of the market increases the need for price changes decreases. Even for markets with 50 buyers two prices result in near optimal revenues. Overall our computations suggest that a simple policy of announcing a small number of prices at the beginning of the sales season is close to optimal. We arrive at this finding assuming that all buyers are strategic. Interestingly Mantrala and Rao (2001) also find that two or three prices changes are adequate. Their analysis, however, assumes that none of the buyers are strategic.

Our analysis is based on several restrictive assumptions, the most significant of which is that buyers have the ability and find it worthwhile to compute their equilibrium strategies. It will be interesting to conduct experiments to see how buyers actually make purchase decisions in such situations. Another assumption we make is that the number of buyers, or the distribution of the number of buyers in the market, is common knowledge. It is quite likely that for many short-life-cycle products the biggest challenge may lie in determining the size of the market. We also assume that all buyers are strategic. We do so to isolate the impact of strategic behavior. In practice only a percentage of the buyers are likely to be strategic. The presence of non-strategic buyers will increase sales at higher prices, and increase the likelihood of stock-outs at lower prices. This in turn should cause the threshold levels for strategic buyers to decrease. The net result could be a mitigation of the role played by strategic buyers.

There are a number of directions in which this work needs to be extended, which include incorporating different types of customers some of whom are strategic and other who are not, incorporating search costs for buyers, and allowing multiple substitutable products.

Appendix. Proofs

[Proof of Lemma 1]

For a buyer with valuation $v \in (P_1, 1]$, let $Z(v)$ denote the change in the expected surplus if the buyer bids in period 2 instead of period 1. Buyers with valuations less than P_1 will not bid in period 1. Recall that $f(p)$ is the probability density function of prices to be charged in period 2. We therefore have $Z(v) = (v - P_1) - \int_0^v (v - p) f(p) dp = (v - P_1) - vF(v) + \int_0^v p f(p) dp$, for any $v \in (P_1, 1)$. Taking the first derivative we get $\frac{\partial Z(v)}{\partial v}|_{v \geq P_1} = 1 - F(v) \geq 0$. Hence $Z(v)$ is an nondecreasing function of $v \in (P_1, 1)$.

Due to the continuity and monotonicity of the function $Z(v)$, if there is an indifference point $v^* \in (P_1, 1)$ such that $Z(v^*) = 0$, then v^* must be the unique. The monotonicity of $Z(.)$ implies that only consumers with valuation $v \geq v^*$ will bid in period 1.

[Proof of Proposition 1] We first consider the case in which there exists a v^* such that $Z(v^*) = 0$. Expected sales in period 1 is $NPr(v \geq v^*) = N(1 - G(v^*))$ and expected sales in period 2 is $NPr(P_2 < v < v^*) = N[G(v^*) - G(P_2)]$. Hence, the expected revenue of the contingent pricing scheme is:

$$E(\pi_R) = N[1 - G(v^*)]P_1 + \int_0^v N[G(v^*) - G(P_2)P_2 f(P_2) dP_2$$

$$\text{subject to} \; : \; (v^* - P_1) - \int_0^{v^*} (v^* - P_2) f(P_2) dp_2 = 0$$

This implies:

$$P_1 = v^* - \int_0^{v^*} (v^* - P_2) f(P_2) dP_2$$

$$E(\pi_R) = N[1 - G(v^*)]v^*[1 - F(v^*)] + [1 - G(v^*)] \int_0^{v^*} P_2 f(P_2) dP_2 +$$

$$\int_0^{v^*} [G(v^*) - G(P_1)]P_1 f(P_1) dP_1$$

$$= N[1 - G(v^*)]v^*[1 - F(v^*)] + \int_0^{v^*} [1 - G(P_1)]P_1 f(P_1) dP_1$$

$$\leq E(\pi_s)[1 - F(v^*)] + E(\pi_s)F(v^*)$$

$$= E(\pi_s)$$

Here, $E(\pi_s)$ is defined as $E(\pi_s) = max_{p \in (0,1)} N[1 - G(p)]p$.

If the indifference point v^* does not exist, then because $Z(.)$ is monotone increasing and $Z(1) < 0$ every buyer would be better off if he postpones bidding until period 2. Therefore, the optimal pricing strategy for the seller is just a single price that maximizes her expected revenues $E(\pi_s)$.

[Proof of Lemma 2] When demand is deterministic, if capacity is limited, there always exists a clearing price P_c that equates supply and demand. Clearly, the set of optimal prices cannot all be lower than P_c nor can they all be above P_c. The range of the optimal prices must include the clearing price P_c.

Let us assume for the sake of a contradiction that the optimal pricing scheme consists of three prices $P_1 > P_2 > P_3$. First, assume that $P_1 > P_c > P_2 > P_3$. In this case, sellers cannot be worse off if they employs a pricing scheme (P_1, P_2) instead of (P_1, P_2, P_3). At price P_2, all inventories can be cleared for sure.

On the other hand, if $P_1 > P_2 > P_c > P_3$, buyers know for sure that at price P_2 there will not be a stock-out. Hence, no sales will occur at price P_1. The seller would be indifferent between (P_2, P_3) and (P_1, P_2, P_3).

By the same logic, we can show that the optimal pricing schemes can never have more than three prices.

[Proof of Proposition 2] A very general policy is for a buyer with valuation v to bid with probability $p(v)$ in period 1. Clearly we must have the following:

a1) $p(v) = 0$ if $v \leq P_1$, and

a2) all buyers with valuations greater than P_2 bid either in period 1 or period 2.

Let $\mathcal{B}$ be the set of values for which $p(v) > 0$. We also require that:

a3) for v distributed as $G(.)$, $Prob(v \in \mathcal{B}) > 0$.

Assume $N - 1$ buyers follow the bidding strategy given above. Let π_1 and π_2 be the probabilities that the Nth buyer gets the product in periods 1 and 2, respectively. Let $v^* \in [P_1, 1]$ be a solution to the equation:

$$(v - P_1)\pi_1 = (v - P_2)\pi_2 \tag{6}$$

Let $Z(v) = (v - P_1)\pi_1 - (v - P_2)\pi_2$. $Z(.)$ represents the incremental value of bidding in period 1 instead of period 2. Since v^* solves equation(6), $Z(v) = (v - v^*)(\pi_1 - \pi_2)$. Due to (a1), (a2), and (a3) $\pi_1 > \pi_2$. Thus, $Z()$ is strictly monotone increasing. Hence, if the valuation of the Nth buyer $v \geq v^*$, then $Z(v) > 0$ and it is optimal for the Nth buyer to bid in period 1. Strict monotonicity of $Z(.)$ also implies that there is at most one solution for equation(6). If equation(6) does not have a solution the Nth buyer will not bid in period 1. What we have just shown is that regardless of the strategies followed by the other $N - 1$ buyers the optimal strategy for the Nth buyer is to bid if his valuation exceeds y^*. This in turn establishes that the unique equilibrium strategy is for all buyers to bid in the first period if and only if their valuations exceed some threshold.

Solutions to equation(1) represent candidate threshold values. If there is only one solution y^* to this equation in the range $[P_1, 1]$, then everyone with a valuation above y^* will bid in period 1. What if there are multiple solutions in the range $[P_1, 1]$? In that case, we let y^* be the smallest solution. Even if $N - 1$ buyers elect to bid if their valuations exceed y^*, it is still attractive for the Nth bidder to bid if his valuation exceeds y^*. Let y' be another solution to equation(1). By definition $y' > y^*$. If some of the buyers choose to bid only if their values are higher than y', then these buyers decrease competition in period 1, making it more attractive for those bidding in period 1. Thus, everyone with valuations above y^* will bid in period 1.

What if there is no solution to equation(1)? Because $Z(P_1) < 0$ and $Z(y)$ are continuous for any $y \in (P_1, 1)$, we know that $Z(y) < 0$ for any $y \in (P_1, 1)$ (otherwise $Z(y) = 0$ would have a solution). In particular, we have $Z(1) < 0$. Thus even if all of the other buyers delay their purchases to the second period, the Nth buyer will also not bid in the first period.

[Proof of Proposition 3]

Proposition 2 establishes the equilibrium for a two-period problem. We use induction on the number of time periods to extend that result to a T period pricing problem.

Assume that for a $T - 1$ period posted pricing scheme with prices $(P_1, \ldots, P_{T-1})$, where $P_i > P_j$ for any $i, j \in \{1, \ldots, T - 1\}$, and $i < j$, buyers' behavior is characterized by a set of thresholds: $y_1^*(k_1) \geq \ldots y_j^*(k_j) \geq \cdots \geq y_{T-1}^*(k_{T-1}) = P_{T-1}$. Only buyers with valuations within the range $[y_j^*(k_j), y_{j-1}^*(k_{j-1})]$ bid for the products at price P_j. The threshold value (y_j^*) depends upon the number of units available for sale (k_j). Let us now consider a T-period problem, and for ease of notation we will denote this additional period as period 0. The T periods pricing scheme is now given by: $(P_0, P_1, \ldots, P_{T-1})$.

Let $\pi_i(y)$ denote the probability that the focal buyer obtains the product given that all buyers follow the policy $[y_1^*(k_1), \ldots, y_i^*(k_i), \ldots, y_{T-1}^*(k_{T-1})]$ and follow the bidding strategy described in Proposition 2 in period 0. As in the proof of Proposition 2, we have $\pi_i(y) \geq \pi_j(y)$ for all $i, j \in \{0, 1, \ldots, T - 1\}$ and $i < j$. Next, consider the following equation:

$$\pi_0(y)(y - P_0) = \max_{j=1,\ldots,T-1} \{\pi_j(y)(y - P_j)\} \tag{7}$$

If equation (7) has a solution $y = y_0^*$, then due to the reasons employed in proposition 2, the focal buyer will buy in period 0 if his valuation $y \geq y_0^*$, otherwise this buyer will delay bidding until a future period.

If equation (7) has no solution in the range of $(P_0, 1]$, no buyer will bid for the product in period 0 as ws also explained in the proof of proposition 2.

Once again, because $\pi_i(y) \geq \pi_j(y)$ if equation(7) has multiple solutions, the smallest solution in the range $[P_0, 1]$ is chosen as the threshold.

Thus, we see that in the first period of a T-periods pricing scheme $(P_0, \ldots, P_{T-1})$, the strategic buyers' equilibrium is also a threshold policy and proposition 2 extends fully to a T-period case.

⌈ Proof of Proposition 4] This proof is similar to that of proposition 3, and we omit the details. If N is uncertain, then we merely have to condition probabilities of a successful bid on N.

⌈ Proof of Proposition 5] Note that the right-hand side (RHS)of (4) is a piece-wise linear function of y for any given y^*. Also, we see that the slope of the left-hand side (LHS) is greater than the slope of RHS for each cut-off value y^*, since $\pi_1(y^*) = \sum_{i=0}^{K-1} Pr_1(i) + \sum_{i=k}^{N-1} Pr_1(i)\frac{K}{i+1} > \sum_{i=0}^{K-1} Pr_1(i)\beta^i$.

Let y^* be the smallest solution in $[P_1, 1]$ for equation (4). For any buyer with a valuation $y > y^*$, the difference in expected surplus between buying in period 1 and buying in period 2, given that all the other buyers follow the policy of threshold value y^*,is : $D(y) = \pi_1(y^*)(y - P_1) - \sum_{i=0}^{K-1} Pr_1(i)\beta^i(y - p_{2r}^*) = (y - y^*)[\pi_1(y^*) - \sum_{i=0}^{K-1} Pr_1(i)\beta^i] > 0$. It is straightforward to see that$D(y) < 0$ for any $y < y^*$. This concludes our proof.

⌈ Proof of Proposition 6] We need to derive the optimal mechanism and show that a T-period posted pricing scheme with P_T converging to v^* yields the same revenues as the optimal mechanism. Given any mechanism, let $m_i(v)$ denote the expected payment by a buyer with a valuation v. The mechanism will also determine probabilities $Q_i(v_i, V_{-i})$. Define $q(v_i) = \int_{V_{-i}} Q(v_i, X_{-i})\mathbf{g}(X_{-i})dX_{-i}$. Due to the revelation principle (Milgrom,2002,1998; and Krishnan, 2002: page 63), expected revenues for any mechanism under equilibrium are given by:

$$E\Pi_Q = \sum_{i \in N} m_i(0) + \sum_{i \in N} \int_{\mathbf{v}} J(v_i)Q_i(v_i, V_{-i})g(v_i, V_{-i})dV \tag{8}$$

Design of an optimal mechanism then becomes one of finding $Q_i(v_i, V_{-i})$ so as to maximize $\mathbf{E}\Pi_Q$, subject to (a) incentive compatibility, (b) individual rationality, and (c) capacity constraints. These constraints in turn can be formulated as follows (Krishnan,2002: page 63):

$$q(v) - q(v') \geq 0 \, for \, all \, v \geq v' \tag{9}$$
$$m_i(0) \leq 0 \tag{10}$$
$$\sum_{i=1}^{N} Q_i(v) \leq K \tag{11}$$

Any allocation scheme that sells the product to buyers with the K highest valuations, provided their valuations are above the threshold v^*, will result in allocation probabilities $Q(v_i, V_{-i}) = 1$ if $v_i > y_i(v_i, V_{-i})$ and 0 otherwise. This allocation scheme ensures that $m_i(0) = 0$ and satisfies the capacity constraint (10). It also maximizes the objective value by allocating unit weights to the K largest $J(v_i)$ for every outcome V, provided $J(v_i) \geq 0$ (See also Krishnan 2002, page 63). The only thing left to show is that the constraint set (8) is satisfied. Observe that if $Q(v_i, V_{-i}) = 1$ for $v_i = v'$, then for all $v_i > v'$, $Q(v_i, V_{-i}) = 1$. This ensures that the constraint set (8) is satisfied and establishes the optimality of the proposed allocation scheme. To complete the proof, we have to show that in a posted price scheme with the limit $T \to \infty$ and in which $P_T \to v*$ buyers with the K

highest valuations get the product, provided that their valuations are above $v*$. Since in the limit P_T equals $v*$, only those with valuations above $v*$ will get the product. All that is left to show is that those with the highest valuations get the product.

Suppose there are K' objects available for sale and the current price is P. In the next "period" the price is going to be $P - \Delta P$. A buyer with valuation y, will buy now if: $\pi(y - P) > (\pi - \Delta\pi)(y - P + \Delta P)$, where π is the probability that the buyer will get the product at price P. Clearly, if the inequality holds for some y', then it holds for all $y \geq y'$. Hence, buyers with higher valuations will bid earlier in the process.

[Proof of Proposition 7] Proof: Let $\theta > 0$ be a large positive number that is used for scaling the market by having θN potential customers and θK units to sell.

For any given pricing scheme $(P_1^\theta, \ldots, P_T^\theta)$, where $1 \geq P_i^\theta > P_{i+1}^\theta$ for $i = 1, \ldots, T - 1$, we know there exist a series of threshold values $y_1^\theta, .. y_{T-1}^\theta$ that characterize customer's purchasing behavior. The number of customers $N(P_i^\theta)$ who bid at price P_i^θ is a random number with mean $N\theta[F(y_{i-1}^\theta) - F(y_i^\theta)]$. By the law of large numbers as $\theta \to \infty$,

$$\frac{N(P_i^\theta)}{\theta} \to N[F(P_{(i-1)r}) - F(P_i r)] \text{ almost surely.}$$

Thus, we have deterministic demand in the limit. Using lemma 2, we conclude that at most two prices are needed. Further, if two prices are needed then the clearing price is located between these two prices.

[Proof of Proposition 8] The proof is based on sample path arguments. For any given realization of value $v_i, i = 1, \ldots, N$, the optimal revenue is

$$R(K) = \sum_{i=1}^{N} J_i(v_i) Q_i(v_i, v_{-i})$$

Now, let us increase the capacity K by one unit. If the $(K+1)^{st}$ largest value in the realization $v_i, i = 1, \ldots, N$ is no greater than the reservation price v^*, then $R(K) = R(K+i), i = 1, \ldots, N - K$. Thus the expected revenue does not change if we increase the supply.

If the $(K+1)^{st}$ largest value v^{K+1} in the realization $v_i, i = 1, \ldots, N$ is greater than the reservation price v^*, then $\Delta R(K) = J(v^{K+1})$. Because of the nature of order statistics and our assumption about increasing virtual value function $J(.)$, the revenue improvement, if positive, will be non-increasing.

Thus, for fixed N expected marginal revenue is a non-increasing, non-negative function of K.

[Proof of Proposition 9] As shown in proposition 8, if we increase the number of units available for sale from K to $K+1$, then revenues will increase provided the valuations of the buyer with the $K+1^{th}$ highest valuation exceeds v^*. By the law of large numbers, the number of customers whose valuations are no less than v^* is asymptotically $N[1 - G(v^*)]$ for sure as $N \to \infty$. In other words, the optimal number of units sold asymptotically approaches $N[1 - G(v^*)]$ as $N \to \infty$.

[Proof of Proposition 10] Let us assume that the firm has inventory levels K_1 and K_2, where $N > K_1 > K_2$, with probabilities α and $(1 - \alpha)$, respectively. If the firm truthfully reveals its inventory levels then the expected revenues are

$$\alpha\Pi(K_1, N, P_1, P_2, y_1) + (1 - \alpha) * \Pi(K_2, N - (K_1 - K_2), P_1, P_2, y_2) \tag{12}$$

On the other hand, if the firm does not reveal its inventory levels, the expected profits are:

$$\alpha\Pi(K_1, N, P_1, P_2, y_\alpha) + (1 - \alpha)\Pi(K_2, N - (K_1 - K_2), P_1, P_2, y_\alpha) \tag{13}$$

It will be optimal for the firm to hide inventories if

$$\alpha\Pi(K_1, N, P_1, P_2, y_\alpha) + (1-\alpha)\Pi(K_2, N - (K_1 - K_2), P_1, P_2, y_\alpha)$$
$$\geq \alpha\Pi(K_1, N, P_1, P_2, y_1) + (1-\alpha)\Pi(K_2, N - (K_1 - K_2), P_1, P_2, y_2) \tag{14}$$

The right hand side of (14) is linear in α, and for $\alpha = 0$ and $\alpha = 1$ the left-hand side and right-hand side are equal. Therefore inequality (14) holds if we can show that the left-hand side is a concave function of α.

Let $H(\alpha) = \alpha\Pi(K_1, N, P_1, P_2, y_\alpha) + (1-\alpha)\Pi(K_2, N - (K_1 - K_2), P_1, P_2, y_\alpha)$. For ease of notation let $\Pi_i = \Pi(K_i, N - (K_1 - K_i), P_1, P_2, y_\alpha)$. We need to show $\frac{d^2 H(\alpha)}{d\alpha^2} \leq 0$. Using the chain rule, we get:

$$\frac{d^2 H(\alpha)}{d\alpha^2} = 2\left[\frac{d\Pi_1}{dy} - \frac{d\Pi_2}{dy}\right]\frac{dy}{d\alpha} + \left[\alpha\frac{d^2\Pi_1}{d^2 y} + (1-\alpha)\frac{d^2\Pi_2}{d^2 y}\right]\left[\frac{dy}{d\alpha}\right]^2 + \left[\alpha\frac{d\Pi_1}{dy} + (1-\alpha)\frac{d\Pi_2}{dy}\right]\frac{d^2 y}{d^2\alpha} \tag{15}$$

As α increases, the probability of higher inventory levels increase; this in turn implies that threshold levels will decrease. Therefore, $\frac{dy}{d\alpha} \leq 0$. Also, as inventory levels increase the loss for the firm from increasing threshold levels is greater (i.e., you are more likely to lose a sale) and thus $\left[\frac{d\Pi_1}{dy} - \frac{d\Pi_2}{dy}\right] \geq 0$. Consequently, the first term on the right-hand side is negative. Since the profit function is assumed to be concave in y, the second term is negative, and the third term is negative because profits decrease with increases in threshold levels and we have assumed that y_α is convex. Because all three terms are negative, the proof is complete.

Endnotes

1. Dynamic pricing is a sub-field for general pricing theory. In pricing theory there is a long tradition of incorporating strategic consumer behavior. In our literature review we elaborate on this point.
2. Filene's basement store in Boston is famous for using this approach (www.filenesbasement.com)
3. Pricing strategies in which number of price changes is random may increase a firm's expected revenues. We ignore that possibility and view the number of price changes as an industry norm.
4. This is one of several allocation mechanisms that maximize revenues.
5. The figure plots $\frac{(posted-contingent)}{posted}$ against K

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