A FACT OF LIFE: PREDICTABLE RETURNS FROM AN EQUILIBRIUM MODEL WITH UNPREDICTABLE FUNDAMENTALS

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ABSTRACT

We show that even when "fundamentals" are *i.i.d.*, a three-period OLG competitive model produces negatively autocorrelated expected returns, market risk premia, and prices; conventional models deliver zero autocorrelation for the same fundamentals. The negative autocorrelation we find arises out of market interactions among consumers that are ex-ante identical but differ by age. This negative autocorrelation is pervasive in our model and persists regardless of the autocorrelation structure of the "fundamentals". Thus the model provides a simple alternative explanation for the observed autocorrelation of returns. A byproduct is low correlation between aggregate consumption and wealth, a puzzle for conventional models.

I. INTRODUCTION

A large number of "puzzles" have been documented in the finance literature in the last two decades, including but not limited to the excess volatility, equity premium and risk-free rate puzzles; these "puzzles" are not necessarily independent. "Puzzles" are defined with respect to the dominant paradigm, in this case the infinitely-lived, representative consumer (ILRC) model. Time variation and predictability of returns have recently become central and controversial. There is also ample evidence that the correlation of wealth and aggregate consumption is far lower than predicted by the ILRC model.

Evidence strongly indicates returns are positively autocorrelated over intervals of less than one year (momentum) and increasingly negatively autocorrelated over longer horizons.[1] This empirical result has been debated extensively. However, in a recent paper Daniels (2003) derives a most-powerful test and shows that most of the early rejections of long-horizon predictability are due to the low power of the tests used. Lettau and Ludvigson (2004) conclude that there is a strong mean-reverting transitory component in wealth, dominated by the stock market. They also show that the correlation between wealth and aggregate consumption is low, because most of the variation in wealth is transitory.

The simple infinitely-lived representative consumer model cannot account for time-varying, negatively autocorrelated returns, nor can it account for the observed low correlation between wealth and consumption. Understandably, many economists are skeptical of empirical findings that cannot be explained with the standard paradigm. Thus, while the empirical evidence was

[1] Summers (1986) shows that returns tend to be negatively correlated over long horizons, and that random walk tests have low power against the alternative hypothesis of a slow mean-reverting component. Fama and French (1988) use Summers' framework to show that predictable expected returns explain long horizon negative correlation. Variance ratio tests by Poterba and Summers (1988) and Lo and MacKinlay (1988) confirm the presence of this stationary component. They find that the negative correlation of returns strengthens with longer horizons. Jegadeesh and Titman (1993) and the momentum studies that followed, confirm the presence of a strong stationary component to asset prices in the near term. Lewellen (2002) finds momentum present even in well-diversified portfolios and argues that *excess covariance* explains momentum profits. Fama (2001) documents that expected equity returns have been declining over the last 50 years.

being debated, a separate strand of the literature attempted to extend the standard model to "explain" the empirical findings. These attempts have not produced a generally-accepted explanation.

We show here that in a simple overlapping generations (OLG) endowments model long-horizon negative autocorrelation of returns is not a puzzle at all; rather it is a robust and natural outcome of the market interactions of utility-maximizing consumers. The model we build is standard in every way, except that consumers live for 3 periods. This fact dramatically changes the properties of asset returns: they display excess volatility, and expected returns, risk premia, and prices are negatively autocorrelated.[2] A byproduct of this negative autocorrelation is a low correlation between wealth and aggregate consumption. This contrasts with the infinite-lived consumer and the 2-generation OLG models, where, under the same conditions, prices and expected returns have zero autocorrelation, their volatility is constant, and consumption and wealth are perfectly correlated.

. There are no frictions, informational asymmetries or other market imperfections in the model.[3] The fundamentals --probabilities, endowments, and dividends-- are time independent and utility functions are CRRA. These fundamentals deliver zero autocorrelation in prices and expected returns in an infinitely-lived representative consumer model as well as a 2-generations competitive OLG model. We also show that the properties of asset returns in this 3-generation OLG model are qualitatively independent of the underlying autocorrelation of the fundamentals.

There is a paucity of alternative theoretical models that convincingly explain return predictability.[4] The standard infinitely-lived representative consumer (ILRC) model has

[2] This model also produces risk premia that are an order of magnitude larger than those produced by the ILRC model. This feature is first documented in Constandinides Donaldson & Mehra (2002).

[3] The *young* are endowed with the only consumption good and when they become *middle-aged* they are further endowed with risky assets that pay dividends the following period. There is no production or storage of the consumption good.

[4] Many variables are found to predict returns, including the price-earnings ratio, dividend yields, and industrial production; most recent is the consumption-to-wealth ratio. See Lettau and Ludvingson (2001).

difficulty generating the discount factor variability necessary for a mean-reverting stationary component in asset prices; aggregate consumption growth is unpredictable and too *smooth* to explain return predictability.[5]

Returns would be autocorrelated in the ILRC model if stock market fundamentals were autocorrelated at the required frequencies.[6] However, the evidence clearly rejects this scenario. Campbell and Shiller (1998), Cochrane (2001), and Fama and French (2001) find that while dividend yields are strong predictors of future returns, dividends themselves lack predictability; the observed properties of "fundamentals" cannot explain return predictability. The consensus is that discount factor volatility rather than dividend volatility accounts for most of the observed returns variation.[7] Le Roy and Porter (1981) and Shiller (1981) show that returns are *too volatile* when the discount factor is assumed constant; at the same time, empirical research documents that returns are predictable. The connection between these observations is that like thunder from lightning, excess volatility arises from return predictability.[8]

Models that do not assume *imperfections* (e.g., frictions, asymmetric information, differences in utility and/or beliefs) or serially dependent fundamentals, introduce hidden state variables and other non-separabilities to engineer predictability. Campbell and Cochrane (1999) model a representative agent with a non-separable power utility. They introduce an unobserved recession state variable, *habit*, into utility; consumption growth is *i.i.d.*[9] *Habit* serves to disconnect the elasticity of substitution from risk aversion. The discount factor is variable and time-dependent because risk aversion increases as consumption approaches *habit* and declines

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[5] See Lettau and Ludvigson (2001).

[6] As discussed in Campbell (2001), the models of Cecchetti, Lam and Mark (1990,1993), Kandel and Stambaugh (1991) require significant predictable changes in dividends to explain equity market volatility.

[7] See Cochrane (2001) and the references therein.

[8] Campbell and Shiller (1988) make this point. The equity premium puzzle is also related to predictability. Fama and French (2001) conclude that the reason the post 1950 equity premium is so high is that prices were *too low* at the beginning of this period. They find the market dividend price ratio is higher at the beginning of the period than at the end unrelated to the stream of dividends. They show that predictability cuts the equity premium roughly in half.

[9] This builds on the work of Abel (1990) and Constantinides (1990).

as consumption moves away from *habit*. Risk premia are negatively autocorrelated over business cycle frequency; they increase in recessions and decrease in expansions.

Constantinides and Duffie (1996) use heterogeneous agents differentiated by permanent idiosyncratic income shocks; there is lack of full insurance. By engineering the patterns of shocks to match desired outcomes, their model can replicate a variety of observed aggregate behavior.[10] They allow partial consumption insurance but design the idiosyncratic shocks so that agents choose to not diversify away the shocks.[11] As in Campbell and Cochrane (1999), the introduction of this state variable in utility disconnects risk aversion and the elasticity of substitution. McCandless and Wallace (1991) create autocorrelation in a perfect foresight OLG model by using utility with satiation. As endowments change, the oscillation between multiple solutions generates negative autocorrelation.

Samuelson (1958) was the first to study the effect of acknowledging that consumers are finite-lived. Overlapping generations models provide a motive for actual trade, absent in representative-consumer models. In a 2-generations OLG model, old consumers sell their assets in exchange for consumption.[12] When consumers live 2 periods, the excess supply of assets is totally inelastic because the *old* (who supply the assets) have no alternative use for them and will sell them regardless of price. Thus, prices play no role in asset allocation except to convince the young to hold all the assets.

By contrast, in a 3-generations model, the *middle-aged* consumers have the opportunity to consume and also rebalance their portfolios. Thus, the excess supply of assets available to the

[10] Cochrane (1991) shows empirical evidence that full consumption insurance, implied by the representative agents' equalizing marginal rates of substitution from state to state, fails in micro data.

[11] This is accomplished by manufacturing high cross-sectional variance in consumption growth when market returns are low. Asset prices support the no-trade equilibrium they engineer. Superimposing a representative agent on an economy with this heterogeneity results in the cross-sectional consumption variance entering the economy-wide intertemporal marginal rate of substitution.

[12] See McCandless and Wallace (1991) and Azariadis (1993) for discussions of non-stochastic OLG models.

young is price-sensitive, which makes returns predictable.[13] We expect such a model to be a better description of asset price behavior than the ILRC model because price-sensitive excess supply of assets is a better representation of markets than price-inelastic excess supply.

The central intuition of our model is as follows. In any state, the *young* can purchase more assets with their endowment and consume more, if asset prices are low. When they become *middle-aged*, their assets pay dividends, leaving them relatively wealthy. They are wealthy enough to finance their consumption today and to keep plenty of the new assets they receive for consumption when they become old. To induce the *middle-aged* to part with some of their assets, the current *young* have to pay high prices. Thus, low prices follow high prices. This property holds regardless of the autocorrelation structure of the underlying "fundamentals," and it results in autocorrelated expected returns and risk premia.

One conclusion is that common perceptions of how markets "should behave", which depend heavily on the properties of ILRC models, do not apply in this more realistic trading environment. It is the combination of multi-period finite lives and the competitive equilibrium among agents with necessarily different horizons that creates these qualitative differences.

One may question if this negative autocorrelation is induced by our stationary binomial specification. After all, in a 2-state time-invariant equilibrium, prices will either stay the same or go up (or down). The example below shows that this structure of uncertainty does not preordain negative autocorrelation in *expected* or *unexpected* returns. Consider a 1-period risky asset in an economy with a time-invariant price distribution. Let the probabilities be state-independent and 0.5. The asset pays $d(g)$ in a g-state and $d(b)$ in a b-state. There will be two prices: $P(g)$ (g-state) and $P(b)$ (b-state), 4 possible returns: $r_{gg} = \frac{d(g)}{P(g)}$, $r_{gb} = \frac{d(b)}{P(g)}$,

[13] The total supply of assets is fixed in our model. The "excess supply" is what the *middle-aged* offer for sale to the *young*.

$r_{bg} = \frac{d(g)}{P(b)}$, $r_{bb} = \frac{d(b)}{P(b)}$, and 2 expected returns, $E(r_g) = 0.5(r_{gg} + r_{gb})$, $E(r_b) = 0.5(r_{bg} + r_{bb})$. In this equilibrium the autocorrelation of prices, *expected* returns, and *unexpected* returns is zero, because the *g*- and *b*-states arrive at random.[14] However, the *unconditional* autocorrelation of observed returns is negative (=-0.50) as a direct consequence of the 2-state nature of the model. But this shows that unconditional autocorrelation of actual returns is the wrong quantity to focus on, because it does not reveal information about the economic processes at work. We study the properties of *expected* returns and risk premia.

The paper contains five additional sections. In section II we state the individuals' maximization problem and the market clearing conditions. In section III we present the solutions and properties of the benchmark models. In section IV we solve our competitive 3-period model, and prove our main results. Section V contains simulation results that show the quantitative importance of our results. It also contains comparisons to the benchmark solutions. Section VI contains concluding remarks.

II. THE MODEL

In this section we describe an endowment-based OLG model in which individuals live 3 periods. The *young* receive a consumption-good endowment and the *middle-aged* receive asset endowments. Assets pay off only in the following period. Uncertainty is generated by two possible states of the world at each date. Both consumption endowments and dividends are high in the *g*-state and low in the *b*-state.

II.1. Specification of the Variables:

States: There are only 2 possible states at any date: A "good" state *g*, and a "bad" state *b*. Arbitrary states are designated as s_{-1}, s, s_{+1}, etc., where

[14] In a stationary model where prices are integrated I(1), prices are positively autocorrelated, but both returns and *expected* returns have zero autocorrelation. The properties of expected returns and risk premia are invariant to this change in specification.

the subscript denotes the date. For example, the sequence ss_{+1} denotes an arbitrary state at date=t, and either of the 2 possible states at the next date, date=$t+1$; $s = g, b$. We refer to a "history", i.e., a specific set of past states as h_{-1}, $h_{-1}s$ or h, as needed; $h_{-1}s \equiv h$.

State Probabilities: Denoted by the states of origin, i, and destination, j: p_{ij}; p_{gg}, p_{gb}, p_{bg}, p_{bb}. These form a time-independent, one-period probability transition matrix.

Generations: There are 3 generations; *young*, *middle-aged*, and *old*, represented by G = 1,2,3.

Population: For simplicity we do not consider population growth in this model. Thus the number of *young* people at date=t are $N(t) = N(0) = 1$, arbitrarily, and total population is 3.

Assets: There are two assets that pay dividends only in the next period. Asset G pays dividends per share $d_G(g) \equiv d_G$ in state g, 0 in state b. Asset B pays 0 dividends in state g and $d_B(b) \equiv d_B$ in state b; $d_G > d_B$.

Asset Prices: $P_k(h,t)$; $k = G, B$.

Portfolio Holdings: The number of shares held by a consumer are $k_G(h,t)$; $k=G, B$.

Endowments: The *young* get a state-dependent but time-independent endowment of the consumption good, $E(g) \equiv E_g$ and $E(b) \equiv E_b$, depending on the state. Each *middle-aged* consumer receives a state-independent endowment of K units of both one-period assets G and B. At date=t, $KN(t-1)$ shares of securities G and B are received as endowments. Endowments and dividends are perfectly correlated, because there are only two states. $E_g > E_b$.

Consumption: Each generation present at date=t consumes $C_G(h,t)$ per capita.

II.2. Uncertainty and Timing

Uncertainty

At date=t, the *young* can be born into one of two possible states, b or g, and they die with one of eight histories at $t+2$: ggg, ggb, gbg, gbb, or bgg, bgb, bbg, bbb. Figure 1 illustrates this binomial structure. All agents know the transition probabilities and the current state.

Asset Markets and Asset Endowments

Under autarky, the *middle-aged* have nothing to eat, because the consumption good is perishable and the share endowments they receive do not yield consumption until the next period. Therefore, in a market economy the *young* purchase shares from the *middle-aged* with some of their consumption endowment. The *young* are net buyers and the *middle-aged* are net sellers of assets. The *old* have no reason to purchase assets in the absence of a bequest motive.

A consumer comes into her middle age with shares purchased when young, and receives K shares of the G and B assets. She consumes the dividends from the shares she brings into the period plus the proceeds from the sale of some of her new share endowment. She holds her remaining shares for consumption in old age. The *old* consume the dividends of the assets they bring into the period. Consumption takes place after dividends are paid and asset markets clear.

After dividends are distributed and asset markets close at date=t, all existing shares must be held by the *young* and the *middle-aged*:[15]

(1) $$k_1(h) + k_2(h) = K, \text{ or } k_1(h) = K - k_2(h); \ \forall\, h.$$

[15] Throughout the paper $x(s)$ denotes a variable that depends only on the current state, while $x(h)$ or $x(h_{-1}s)$ denotes a variable that potentially depends on history as well as the current state.

Consumption, Consumption Endowments, and the Goods Markets

The total supply of consumption at date=*t* consists of the total goods endowment the young receive, plus the dividends from the total shares that were issued at date=*t-1*:

(2) $$K\left[d_G(s)+d_B(s)\right]+E(s),\ \ s=g,\ b.$$

Each member of the *young*, *middle-aged* and *old* generations consumes $C_1(h)$, $C_2(h)$, and $C_3(h)$, respectively. Total demand for consumption then is,

(3) $$C_1(h)+C_2(h)+C_3(h).$$

II.3. The Individual's Maximization Problem

Statement of the Problem

Everyone possesses the same time-separable CRRA utility function, $U(C)=\dfrac{C^{1-g}}{1-g}$, where g is the relative risk aversion. The individual born at date=*t* chooses a lifetime consumption plan that maximizes expected lifetime utility, conditional on rationally anticipated market prices, and possibly the state history $h_{-1}s \equiv h$:

(4) $$\underset{\left\{\begin{matrix} C_1(s), G_1(s), B_1(s) \\ C_2(sh_{+1}), G_2(sh_{+1}), \\ B_2(sh_{+1}), C_3(sh_{+1}h_{+2}) \end{matrix}\right\}}{\text{Max}} E_t\left\{W\left(1, hh_{+1}h_{+2}\right)\right\} = \quad U\left[C_1(h)\right]+\mathrm{E}_t\left\{dU\left[C_2(h_{+1})\right]+d^2U\left[C_3(h_{+2})\right]\right\},$$

subject to the age=1, 2, 3 budget constraints we discuss below. The following budget constraints must also be satisfied:

The *young* (they must buy assets):

(5) $$C_1(h_{-1}s)=E(s)-P_G(h_{-1}s)G_1(h_{-1}s)-P_B(h_{-1}s)B_1(h_{-1}s).$$

The *middle-aged* (they receive asset endowments and must keep some for old age):

(6) $$C_2(h_{-1}s)=G_1(h_{-1})d_G(s)+B_1(h_{-1})d_B(s)+P_G(h_{-1}s)\left[K-G_2(h_{-1}s)\right]+P_B(h_{-1}s)\left[K-B_2(h_{-1}s)\right]$$

The *old* (they consume all their resources):

(7) $$C_3(h_{-1}s)=G_2(h_{-1})d_G(s)+B_2(h_{-1})d_B(s).$$

First-Order Conditions

The full derivation of the first order conditions is in Appendix 1. Below we present a short summary. The *young* and *middle-aged*, equate their expected marginal rates of substitution, because they trade in the securities market,

(8) $$q_{h_{-1}ss_{+1}} = dp_{ss_{+1}}\left[\frac{C_2(h_{-1}ss_{+1})}{C_1(h_{-1}s)}\right]^{-g} = dp_{ss_{+1}}\left[\frac{C_3(h_{-1}ss_{+1})}{C_2(h_{-1}s)}\right]^{-g};\ \forall\ h,\ s.$$

The *old* have no relevant marginal rate of substitution.

Equilibrium at Date=t

The *young*, *middle-aged* and *old* jointly determine equilibrium allocations. In addition to the constraints associated with the individuals' maximization problems, market-clearing conditions must be satisfied at every date. These are:

1. Asset markets:

At the end of date=*t*, all shares outstanding are held by the *young* and the *middle-aged*,

(9a,b) $$G_1(h_{-1}s) = K - G_2(h_{-1}s), \qquad B_1(h_{-1}s) = K - B_2(h_{-1}s);\ \forall\ h,\ s.$$

Asset prices must satisfy the present value relation,

(10a,b) $$P_G(h) = q_{hg}d_G, \qquad P_B(h) = q_{hb}d_B.$$

2. Goods markets:

Demand for consumption (equation 3) and its supply (equation 2) must be equal,

(11) $$C_1(h_{-1}s) + C_2(h_{-1}s) + C_3(h_{-1}s) = E(s) + K[d_G(s) + d_B(s)].$$

This last condition is redundant.

I. SOLUTIONS AND PROPERTIES OF BENCHMARK MODELS

We accomplish two tasks in this section. The first is to provide solutions for three standard models that we use as benchmarks for comparison --the ILRC, the competitive 2-generations

OLG, and social planner 3-generations OLG. We choose these three models as benchmarks for the following reasons. The ILRC model is the most common model in the finance literature, and its properties are well known and understood. The competitive 2-generations OLG model is the most popular model in its class, and it captures the effects of rudimentary trade in assets. These two models exhibit the generally agreed-upon properties of prices in efficient markets. Finally, 3-generations general equilibrium OLG models are rarely used. The social planner solution to a 3-generations OLG model provides insight into the effect of just extending consumers' lives for one period.[16] Comparing the competitive market 3-period OLG solution to these three benchmark models helps us gauge separately the effect of finite life, of adding a period to the 2-generations model, and the effect of competition on the properties of asset prices.

The second task is to show the asset price and return properties of these models. These three benchmark models have many "typical" properties in common: their solutions are periodic, time-independent, and they reflect closely the properties of the underlying endowment process. Consequently, return volatility is time-independent and expected returns and prices are *i.i.d.*, in all three models. The benchmarks help show the extent to which the properties of our competitive 3 OLG model (high negative autocorrelation of prices and expected and realized returns, and excess and time-dependent volatility in asset returns) deviate from the properties of the benchmarks.

In the discussion of the properties that follows, we assume that probabilities are state-independent, i.e., $p_{gg} = p_{gb}$, $p_{bg} = p_{bb}$.

[16] An exception is Constandinides, Donaldson, and Mehra (2002). They construct a 3-period OLG model, to study the effect of borrowing constraints on equity premia. They also compute a numerical solution.

III.1. Solutions of the Benchmark Models

The solution of the ILRC model is in Appendix 2A. The equilibrium is time-independent and periodic; the consumption allocations for any g- and b-state are unique. Asset prices are given by,

(12a,b) $$P_G(g) = dp_{gg} d_G, \quad P_G(b) = dp_{bg} ?^{-g} d_G,$$

(12c,d) $$P_B(b) = dp_{bb} d_B, \quad P_B(g) = dp_{gb} ?^{g} d_B, \text{ where, } ? = \frac{E_G + Kd_G}{E_B + Kd_B} > 1.$$

The solution of the competitive market 2-OLG model is in Appendix 2B. The equilibrium is again periodic and time-independent; it consists of a set of four nonlinear equations in the four consumptions --$C_1(g)$, $C_1(b)$, $C_2(g)$, $C_2(b)$-- that must be solved numerically. Consumption allocations are unique for each state type (g, b), and state histories are irrelevant. Thus there are only four possible state prices, two associated with g-states and two with b-states -- $q_{gg}, q_{gb}, q_{bg}, q_{bb}$. Asset prices are,

(13a,b) $$P_G(g) = q_{gg} d_G, \quad P_G(b) = q_{bg} d_G,$$

(13c,d) $$P_B(g) = q_{gb} d_B, \quad P_B(b) = q_{bb} d_B.$$

The solution of the 3 OLG Social Planner model is in Appendix 2C. The 3 OLG SP solution shares many of the properties of the above benchmarks. The equilibrium is once again periodic and time-independent.[17] Consumption allocations are unique for each of the two states, and state histories are irrelevant. The solution for the "shadow" state prices in this economy is,

(14a,b) $$P_G(g) = p_{gg} d_G, \quad P_G(b) = p_{bg} ?^{-g} d_G,$$

(15a,b) $$P_B(b) = p_{bb} d_B, \quad P_B(g) = p_{gb} ?^{g} d_B, \text{ where again, } ? = \frac{E_G + Kd_G}{E_B + Kd_B} > 1.$$

[17] When state probabilities are state-independent there are four state prices but only two possible configurations of total resources. Thus consumption, interest rates, risk premia, and asset prices depend only on the current state; they each have two values, one for *good* states and one for *bad* states.

These asset prices are identical to those of the representative-consumer equilibrium but for the absence of the time preference factor, d.

In all three models, prices, expected returns and risk premia have zero autocorrelation. We provide simulation examples for these models in section V.

III.2. Properties of Asset Prices and Returns of the Benchmark Models

A common characteristic of our three benchmark models is that their solutions depend only on the current state and not on history; there are two equilibrium allocations, one for each state. In state s, (s = g, b), asset prices are $P_G(s)$ and $P_B(s)$, and the value of the market is, $M_s K = [P_G(s) + P_B(s)]K$. There are four possible realized market returns: $r(ss_{+1}) = \frac{d_{s_{+1}}}{M_s} - 1$, and there are only two possible expected returns, one for each state: $Er(s) = \frac{Ediv}{M_s} - 1$, where $Ediv \equiv p_g d_G + p_b d_B$.

The variance of expected returns is constant, $s_{Er}^2 = p_g p_b \left[Ediv \left(\frac{1}{M_g} - \frac{1}{M_b} \right) \right]^2$. Note that this variance depends directly on the dispersion of the market price and only indirectly on the dispersion of dividends. The autocorrelation of expected returns is zero. This is also true for each asset price and the price of the market.[18]

An important conclusion is that return and price volatility are related to both dividend and expected return (discount factor) volatility. If the discount factor were constant, then prices would be constant, regardless of dividend volatility, because the product of state prices and the state-dependent dividends would always be the same. The volatility of *realized* returns would depend only on the volatility of dividends. But even in the simplest stationary model, state

prices must be state-dependent and thus prices must be volatile.[19] We will refer to the volatility these benchmark models exhibit as "fundamental" but with the clear understanding that it is not a simple outcome of dividend volatility alone.[20]

II. THE COMPETITIVE 3 OLG EQUILIBRIUM AND THE PROPERTIES OF ASSET PRICES

Balasko, Cass, and Shell (1980) show that a competitive solution exists for all OLG endowment economies. In this section we present the solution of the model, show that it is unique, and derive the properties of asset prices and returns. We start by proving three theorems that are necessary to construct the solution of the competitive market 3 OLG model.

IV.1. Three Theorems

Theorem 1: The social planner solution is not attainable in a competitive market.

Proof: In Appendix 3 we show that the competitive solution is determined by both the time=t resource constraint and the individual consumer's lifetime budget constraint. These constraints cannot be made identical under uncertainty, thus the social planner solution is not feasible.[21]

The first-best solution fails to obtain. As shown in Chattopadhyay and Gottardi (1999), the fundamental reason for this failure is that future generations are not present at time t=0, to engage in appropriate contracts that would result in the first-best solution. A first-best solution is characterized by full risk sharing by all generations. Full risk sharing is incompatible with

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[18] The autocorrelation is zero only when probabilities are state-independent; $p_{gg} = p_{bg}$.

[19] Dividend volatility has an indirect effect on price volatility. To the extent that dividends contribute to consumption, their volatility helps determine the dispersion of state prices, and through that the volatility of prices and returns.

[20] In contrast to the stationary case, it is easy to construct a binomial growth model in which there are only two state prices and the price and return volatility is related to dividend volatility and the growth rate (see Benninga & Protopapadakis 1991).

individuals' budget constraints unless all agents are present at the creation, and are resurrected sequentially. Thus the competitive solution is necessarily second-best but unique, as we show in Theorem 4 in the next section.

Theorem 2: The competitive solution in this economy is aperiodic, even though resource allocations are *i.i.d*.

Proof: In Appendix 4, we show that a solution of any fixed periodicity is reduced to the social planner solution, which is unattainable by Theorem 1.

The intuition behind this result is that the *middle-aged* come into the period with a wealth position determined at the previous date. As a result, the *middle-aged* bring history into the current period. Even though the resource availability and payoffs are the same (for all *g*- or all *b*-states), the excess supply of assets by the *middle-aged* depends on the shares they bring into the period. Furthermore, the *young* and the *middle-aged* have different maximization problems, because their lifespans differ. Therefore, resources are valued differently and shared differently at different dates, even when the fundamentals are the same.

Lemma 1: The ratio of consumptions between the *middle-aged* and *old* is fixed by their decisions in the prior period, regardless of the state that obtains.

Proof: This follows directly from the first order conditions:

(14a) $$q_{hs} = dp_{ss_{+1}}\left[\frac{C_1(h)}{C_2(hs)}\right]^g = dp_{ss_{+1}}\left[\frac{C_2(h)}{C_3(hs)}\right]^g \;;\; s = g, b.$$

Simplifying yields:

(14b) $$\frac{C_1(h)}{C_2(hs)} = \frac{C_2(h)}{C_3(hs)}, \text{ and finally, } \frac{C_2(h)}{C_1(h)} = \frac{C_3(hs)}{C_2(hs)} \equiv R(t)\,;\; s = g, b.$$

Time=*t* time=*t+1* Q.E.D

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[21] Under certainty, there is a "state price" for which the individual's wealth constraint is identical to the economy-wide resource constraint, and the SP solution is one of the two solutions to the model (see

Theorem 3: $R(t)$, as defined in Lemma 1, fully characterizes the equilibrium, given endowments and history, i.e., $R(t\text{-}1)$.

Proof: Let "*" denote one equilibrium allocation. Are there other equilibria in which $R(t) = R^*(t)$? The strategy is to show that this implies a contradiction.

(15a) $$\frac{C_2^*(h,t)}{C_1^*(h,t)} = \frac{C_3^*(hg,t+1)}{C_2^*(hg,t+1)} = \frac{C_3^*(hb,t+1)}{C_2^*(hb,t+1)} \equiv R^*(t), \text{ and}$$

(15b) $$\frac{j\,C_2^*(h,t)}{j\,C_1^*(h,t)} = \frac{wC_3^*(hg,t+1)}{wC_2^*(hg,t+1)} = \frac{y\,C_3^*(hb,t+1)}{y\,C_2^*(hb,t+1)} \equiv R(t) = R^*(t),$$

where j, w, y are positive constants.

The date=t consumption ratios are optimally set at date=$t\text{-}1$. This implies that $\frac{C_3^*(h,t)}{C_2^*(h,t)} = R^*(t-1)$. It follows that, $C_3^*(h,t)$ is determined at date=$t\text{-}1$.

The total consumption implied by this candidate equilibrium at date=t then is,

$$C_1^*(h) + C_2^*(h) + C_3^*(h) = j\,C_1^*(h) + j\,C_2^*(h) + C_3^*(h) = \text{Constant}\ .$$

But since total available consumption is fixed, the only possible equilibrium with the scaled C_1 and C_2 is at $j = 1$. Thus, $R(t)$ describes a unique equilibrium. Q.E.D.

$R(t)$ plays a key role in the solution that follows.

IV.2. The Solution Equations

The solution involves two recursive equations in the shares held by the *young* and in the ratio of consumptions, $R(t)$. These equations give the solution for every node on the binomial tree. We modify our state notation somewhat to improve clarity. *Good* states (g) are *even* ($2i$) and *bad*

Azariadis, 2002).

states (b) are *odd* ($2i$+1). For any state i, the successor states in the binomial tree are $2i$, and $2i$+1 (see Figure 1); the parent state of any state is Integer(i/2). We retain the g, b notation for the history-independent variables, probabilities, endowments and dividends. The details of the solution are in Appendix 5. The equations for the two states are:

$$(16)\quad R(i) = R^g(i) = \frac{K - G_1(i)}{\frac{E_G}{d_G} + G_1(i)}\left(\frac{1+R(2i)}{R(2i)}\right) = R^b(i) = \frac{K - B_1(i)}{\frac{E_B}{d_B} + B_1(i)}\left(\frac{1+R(2i+1)}{R(2i+1)}\right),$$

$$(17a)\qquad \frac{E_g - C_1(2i)}{C_1(2i)^g} = d\,R(2i)^g \begin{bmatrix} p_{gg} d_G^{\,1-g}\left(K - G_1(2i)\right)^{-g} G_1(2i) + \\ p_{gb} d_B^{\,1-g}\left(K - B_1(2i)\right)^{-g} B_1(2i) \end{bmatrix},$$

where, $C_1(2i) = \left[R(2i)+1\right]^{-1}\left[d_G G_1(i) + E_g\right]$, and,

$$(17b)\qquad \frac{E_b - C_1(2i+1)}{C_1(2i+1)^g} = d\,R(2i+1)^g \begin{bmatrix} p_{bg} d_G^{\,1-g}\left(K - G_1(2i+1)\right)^{-g} G_1(2i+1) + \\ p_{bb} d_B^{\,1-g}\left(K - B_1(2i+1)\right)^{-g} B_1(2i+1) \end{bmatrix},$$

where, $C_1(2i+1) = \left(R(2i+1)+1\right)^{-1}\left(d_B B_1(i) + E_b\right)$, and,

$$(17c)\qquad R^g(i) \equiv \frac{C_3(2i)}{C_2(2i)} = R^b(i) \equiv \frac{C_3(2i+1)}{C_2(2i+1)} = R(i) \equiv \frac{C_2(i)}{C_1(i)}$$

We now show that the solution is unique.

Theorem 4: The equilibrium is unique given initial conditions and endowment structure.

Proof: Consider the difference equation of our solution (equation 16).

$$R(i) = R^g(i) = \frac{K - G_1(i)}{\frac{E(g)}{d_G(g)} + G_1(i)}\left(\frac{1+R(2i)}{R(2i)}\right) = R^b(i) = \frac{K - B_1(i)}{\frac{E(b)}{d_B(b)} + B_1(i)}\left(\frac{1+R(2i+1)}{R(2i+1)}\right).$$

The equation is of the form,

$$(18)\qquad R(i) = \Psi(i)\frac{1+R(2i)}{R(2i)} = \Psi(i)\left[1 + \frac{1}{R(2i)}\right] = \Xi(i)\left[1 + \frac{1}{R(2i+1)}\right],$$

where $\Psi(i)$, $\Xi(i) > 0$ denote values known at state i (parameters and variables). Thus, given history (state i), the future Rs --$R(2i)$ and $R(2i+1)$-- are unique. From Theorem 3, $R(t)$ characterizes a unique equilibrium, given history. Therefore, the equilibrium is unique.[22]

QED.

IV.3. Properties of Asset Prices and Returns

In this section we show that asset prices and returns are negatively autocorrelated; this contrasts sharply with their properties in the benchmark models. Date=t returns (for state g or b) depend only on date=t-1 prices in all these models, since date=t payoffs are fixed by state. Therefore, for simplicity we examine the properties of prices rather than returns.

Theorem 5: *R(t)* is negatively autocorrelated.

Proof:

Consider again the solution equations, $R(i) = \Psi(i)\left[1 + \frac{1}{R(2i)}\right] = \Xi(i)\left[1 + \frac{1}{R(2i+1)}\right]$, where $\Psi(i) \equiv \frac{K - G_1(i)}{\frac{E(g)}{d_G(g)} + G_1(i)}$, and $\Xi(i) \equiv \frac{K - B_1(i)}{\frac{E(b)}{d_B(b)} + B_1(i)}$. Both quantities are small when G_1 and B_1 are high. C_3 is predetermined in state=i by shares purchased the previous period, so that C_1 and C_2 are negatively related. When $R(i)$ is smaller-than-average, C_1 is larger- and C_2 is smaller-than-average, by definition. A larger-than-average C_1 implies larger-than-average B_1 and G_1, because consumption is a normal good.[23]

Thus, when $R(i)$ is smaller-than-average, $\Psi(i)$ and $\Xi(i)$ are smaller-than-average, and $R(2i)$ must be larger-than-average. The reverse is true when $R(i)$ is larger-than-average.[24] QED.

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[22] If the economy starts with 3 coexistent generations, the asset holding of the young have to be given from outside the model. If the economy self-starts (1 generation the first period, etc.) this specification of the genesis constitute the initial conditions.

[23] C_1 is large because share prices are low. Thus, the *young* are able to purchase both more current consumption and more shares for more consumption is the next period.

[24] An alternative way to understand this process is to consider a higher q_{sg}, one of two current state prices. The *young* are less well off because they are net purchasers of period-2 consumption, while the *middle*-

Lemma 2: State prices and therefore asset prices are negatively autocorrelated regardless of the autocorrelation structure of the underlying dividends and endowments.

Proof: A formal proof is in Appendix 6. Here we provide the fundamental economic intuition of the lemma. Consider an equilibrium with high $R(t)$ (C_2 is high and C_1 is low). High C_2 means that the *middle-aged* are "well off"; they have brought in sufficient shares from the previous period, and will save enough to have high consumption in the following period. This implies that they will sell relatively few of their endowed shares, and at high prices. The *young* have low consumption because share prices are high and their goods endowment buys relatively few shares.

In the following period the situation is exactly reversed. The poor *young* of the previous period are now the poor *middle-aged*. The *middle-aged* have brought in few shares from the previous period, they will save relatively small amounts and will have low consumption now and in the following period; C_2 is low. This implies that they will sell much of their share endowment, and at low prices. But this means that the *young* can have high consumption because share prices are low and their goods endowment buys relatively more shares.

The negative autocorrelation of asset prices contrasts sharply with the benchmark models with the same underlying fundamentals, in which asset prices are *i.i.d*. History affects today's solution, unlike the benchmark models, because the *middle-aged* consumers have an elastic excess supply of assets. The time series properties of the 3 OLG competitive equilibrium no longer resemble those of the benchmark models.

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aged are better off. Since all consumptions are normal goods, the *youngs'* current consumption, $C_1(s)$, will be lower. This implies that the consumption of the *middle-aged* must be higher, because the *old* are unaffected by this higher price. Thus $C_2(s)$ is higher. From the definition, $R(t) = C_2(s)/C_1(s)$, $R(t)$ is higher unambiguously.

V. SIMULATION RESULTS

In this section we examine the numerical importance of the autocorrelation properties proved in the section IV, by comparing the numerical solution of our model to the three benchmark models.

First we discuss briefly the numerical method we use to compute the solution. We compute the solution in two steps. It is clear by inspection of equation (16) that these functions are independent of state and time (though the solution is not). The first step is to use the two difference equations (16) and (17) to compute iteratively the two functions that map $R(i)$ to the shares of the young $G_1(i)$, $B_1(i)$.[25] We do this by specifying $0.05 \leq R(i) \leq 30$ in increments of 0.001, and iterating until the functions converge.[26]

The second step is to select a random path 35,000 periods long, taking into account unequal probabilities when relevant. We then construct the sequence of equilibria along this path by using the function we computed. We eliminate the first 5,000 observations to eliminate the effect of the starting shares, so that our sample path has 30,000 observations. We compute the numerical properties of the model from this sample.

V.1. Parameter Values

We make no attempt to calibrate our economy to any actual economy. In any case, we have already shown that the autocorrelation property of our model is not dependent on the properties of the fundamentals. We assign generally accepted values to parameters such as relative risk aversion and time preference. However, we choose the ratio of dividends-to-endowments and the variance of dividends and endowments so as to obtain positive interest rates on average and

[25] We solve our model with $E_g/d_g = E_b/d_b$. If these ratios differ across states then four instead of two functions would be required to describe the equilibrium.

[26] This function is clearly time independent, and it embodies all the restrictions of the model. The numerical procedure it very stable over a wide range of parameter values.

significant risk premia. We consider a "period" in our model to correspond to 20 years and report annualized figures for returns.

The values of all the model parameters for the baseline simulations are as shown below.[27]

RRA	$g = 5.00$
Time Preference	$b = 0.99^{20} = 0.8179$
Transition Probabilities	$p_{gg} = p_{bg} = 0.50$; $p_{bb} = p_{gb} = 0.50$.
Endowments	$E_g = 13.66$; $E_b = 10.10$; $\bar{E} = 11.88$.
Dividends	$d_G(g) = 6.37$; $d_G(b) = 0.00$; $d_B(g) = 0.00$; $d_B(b) = 4.71$.
Number of Shares	$K = K_0 = 3.00$

V.2. Quantitative Properties of the Benchmark Models

We can loosely characterize the return properties of the benchmark models as being consistent with properties that we normally associate with "rational" and "efficient" markets. These characteristics include the notion that the properties of expected returns should reflect the properties of dividends; for example, *i.i.d.* dividends should produce *i.i.d.* expected returns.[28] Table 1 shows the characteristics of important variables of our three benchmarks: the 2 OLG competitive, the representative-consumer, and the 3 OLG social planner solutions.

The expected market returns, ER_m, as well as the market risk premia, ER_p, of the 2 OLG solution are substantial and positive, their standard deviations are state- and time-independent, and their autocorrelations are zero.[29] The *B*-asset (that pays lower dividends in *bad* states) has

[27] We note any changes in these values when the need arises. Including or eliminating the first 5,000 observations or changing initial conditions does not affect the time series properties we report.

[28] However, we have already shown that in the benchmark models return volatility does not depend only on dividend volatility.

[29] The large expected returns are the result of high risk-free rates and relatively small but positive market risk premia. Compared to the empirical facts for the U.S., these premia are still "small", much more so in the

a higher price, P_B, than the G-asset, P_G. The two assets have the same coefficients of variation, and their prices are not state-dependent.

The representative consumer model solution has broadly similar characteristics. The average ER_m is positive and very small, and it is negative in the g-state. The market risk premia are very small but positive in both states.[30] Compared to the 2 OLG solution, the standard deviation of ER_m is slightly higher but that of ER_p is considerably smaller. Like the 2 OLG solution, consumption increases with age in both good and bad states, and $P_B > P_G$. The coefficient of variation is the same for both assets and in both states, and it is higher than in the 2 OLG model. Finally, the autocorrelations of expected returns, risk premia, and prices are zero.

The 3 OLG SP solution has some ordinary and some peculiar features. Because the state prices are those of the representative consumer model divided by the time preference parameter, the returns properties are very similar. The ER_ms are slightly smaller than in the representative consumer solution, and they are negative on average.[31] Asset prices are slightly higher and risk premia slightly lower. The standard deviation of ER_m and the coefficient of variation of the prices are again time- and state-independent, and the autocorrelations of expected returns, risk premia, and prices are zero.

The volatility and autocorrelation features in the three benchmark models are very similar. There is little difference between the OLG social planner and the representative consumer

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representative consumer and 3 OLG SP solutions. This is a reflection of the "equity-premium puzzle": the CRRA utility function and the modest RRA value.

[30] The very small average premium is the reason for the very high Coefficients of Variation.

[31] These are only "shadow" prices on which trades do not occur. The asset prices are inferred from the marginal rates of substitution, because consumers do not have the opportunity to trade. An interesting property of the solution is that consumption declines with age. A time preference less than 1.0 implies that consumers prefer consumption today to tomorrow. The economics of the infinitely-lived representative consumer model causes the interest rate to be positive, on average, in order to persuade consumers to defer consumption just enough to balance out demand and supply in each period, even with no growth. In the 3 OLG model the social planner has another, better, possibility. She distributes the consumption to those that

solutions in this respect. It is *not* finite life per se but the competitive equilibrium among agents with different horizons that creates qualitative differences in Table 1. Below, we compare the 3 OLG competitive solution results to the "typical" benchmark model properties, rather than to each one separately.

V.3. Properties of the 3-OLG Competitive Solution

V.3.a Comparison of the 3 OLG Competitive Solution With the Benchmarks

Table 2 shows average values as well as the volatility properties of expected returns, ER_m, the expected market risk premia, ER_p, and prices, P_G and P_B for the 3 OLG competitive solution. The average values of the variables are closer to the 2 OLG competitive solution than the other two benchmarks. It is interesting to note that the market risk premia for both the 2- and 3 OLG competitive solutions are an order of magnitude larger than those of the representative consumer model, for the identical RRA and aggregate risk.[32] The standard deviations of ER_m and ER_p are generally more than double their counterparts in the benchmark models. Furthermore, they are time-varying. For example, ER_m's standard deviation for the ***g*** to ***g*** transitions is 12.92%, but it is 11.02% for the ***b*** to ***g*** transitions. These patterns hold for all the variables we examine. The coefficients of variation of P_G and P_B are also much larger than their counterparts in the benchmark models, and they also exhibit time variation. These results show that the combination of a 3rd generation and a competitive market results in major qualitative differences in asset price properties compared to the benchmarks.

As we discuss earlier, the cause of the "excess" and time-varying volatility is the negative autocorrelation of prices and returns, to which we turn next.

V.3.b Negative Autocorrelation of Expected Returns, Risk Premia, and Prices

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value it most, which are the *young*, the *middle-aged*, and the *old*, in that order. Thus the *old* receive the least consumption because they obtain the least utility from it (because of positive time preference).

[32] The "equity premium puzzle" is much less acute in OLG models. This is studied by Constandinides et. al. (2002). In Table 1 ER_p is larger in the 2 OLG competitive solution compared to the other benchmarks.

Theorem 5 and Lemma 2 show that a fundamental characteristic of the 3 OLG competitive solution is the negative autocorrelation of asset returns and prices. Figures 2A and 2B illustrate this property for the phase diagrams for $R(t)$ and ER_m. The phase diagram for $R(t)$ is negatively sloped as proved in Theorem 5. Here there are 4 distinct functions, by state of origin and destination; each function is labeled accordingly. Figures 2C and 2D show the phase diagrams for ER_p and P_b; all have negative slopes.

Table 3, Panel A shows the 1st order autocorrelations for ER_m, ER_p, P_G and P_B. The 1st order autocorrelations are highly negative for all four variables. The simulation shows that the negative autocorrelation is numerically very important.

Figures 3A and B illustrate the mechanism that makes Lemma 2 hold. Figure 3A shows the negative relation between asset prices (only P_B is shown) and the *youngs'* consumption, while Figure 3B shows the positive relation between the consumption of the *middle-aged* and the same asset prices. High asset prices make the *middle-aged* well off and the *young* worse off. In the following period, the "poor" *middle-aged* (previously *young*) have low demand for assets and asset prices are low.

The market risk premium, ER_p, is negatively autocorrelated as well, because it varies inversely with asset prices. When prices are high ER_p is low, and when prices are low ER_p is high. There are two complementary reasons for this. For equal probabilities, ER_p can be expressed as $ER_p = \frac{1}{q_b}\left[\frac{0.5(1+f)(1+q)-(1+qf)}{(1+qf)(1+f)}\right] > 0$, where $q \equiv \frac{d_G}{d_B} > 1$, and $f \equiv \frac{q_g}{q_b} < 1$ because of risk aversion. It is clear that if f and q are fixed, a proportional increase in the state prices (and therefore asset prices) decreases ER_p. The second reason has to do with risk management by the two market-active generations. The young face more risk than the *middle-aged*, because their share endowments are subject to price risk in the next period. Therefore, they hold portfolios less risky than the market and the *middle-aged* hold portfolios that are riskier than the

market.[33] At low prices and a fixed risk premium, the *young* wish to increase their holdings while keeping their portfolio composition the same. But this attempt by the *young* requires the *middle-aged* to hold even riskier portfolios![34] The result in equilibrium is a higher price of risk.

This negative autocorrelation of returns, risk premia, and prices is a fundamental characteristic of the 3-period OLG competitive model. It arises from the market interaction of agents, which are ex-ante identical but whose wealth position differs because of their histories and their horizons. In the next section we explore how this negative autocorrelation of returns might be modified if transition probabilities are state-dependent.

V.3c. The Effect of Negative Underlying Probabilities on the Autocorrelation of Returns

We explore the effect of changing the autocorrelation structure of dividends on the autocorrelation of expected returns and prices. Lemma 2 asserts that prices will be negatively autocorrelated, without reference to the autocorrelation properties of dividends and endowments. However, the magnitude of this negative autocorrelation may depend on the dividend autocorrelation. In Table 3, Panels B and C we report two experiments, one with positive and one with negative autocorrelation in the dividend process: $p_{gg} = p_{bb} = 0.8$, and $p_{gg} = p_{bb} = 0.2$, respectively.

The autocorrelation of dividends has only a minor influence on the autocorrelation of prices and returns. Positive autocorrelation of dividends makes ER_m and ER_p somewhat *more* negatively autocorrelated, while negative autocorrelation of dividends makes it *less* negatively autocorrelated. The negative autocorrelations of ER_m and ER_p are substantial in all three cases, even though the dividend autocorrelation changes drastically. The reaction of prices is mixed. P_G is slightly *less* negatively autocorrelated while P_B is substantially *more* negatively autocorrelated when dividends are positively autocorrelated.

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[33] We found no counter-example in our simulations.

[34] An example may be helpful: Let the *young* hold 1.0 share of the *G-asset*, while their portfolio composition is *G1/B1* = 0.83 (a typical value). This implies that *G2/B2* = 1.11. If the *young* buy 10% more shares and maintain their portfolio composition, the *middle-aged* would have to hold *G2/B2* = 1.14.

The important conclusion from this experiment is that the substantial negative autocorrelations of returns and prices that we found is driven by the workings of the market and not by the structure of the fundamentals. The negative autocorrelation remains, regardless of the autocorrelation of the fundamentals.

V.3d. The Correlation of Wealth with Aggregate Consumption

Table 4 shows the correlation of aggregate consumption as well as the consumption of each generation with wealth.[35] The 1st column shows that aggregate consumption has a low correlation with wealth. The succeeding columns show the reason for this low correlation: the correlation of consumption with wealth varies by generation. This result stands in sharp contrast with our benchmark models, where aggregate consumption is perfectly correlated with wealth.

The low correlation between wealth and aggregate consumption is generated by the negative autocorrelation of asset prices and thus wealth. High asset prices today imply high consumption for the *middle-aged* but low consumption for the *young* who have to acquire the high-priced assets. The *old* are not affected by the current market conditions; but because of the negative correlation, prices were likely to have been low when they were *middle-aged*, which implies low consumption for them now.

VI. CONCLUSION

The contribution of this paper is to show that in a 3-generation OLG competitive equilibrium, expected market returns, risk premia, and asset prices are negatively autocorrelated, regardless of the autocorrelation structure of the fundamentals. Market interactions between identical consumers, except for their age, gives rise to complex price and return dynamics even when dividends, endowments, and aggregate consumption are *i.i.d*. This result is qualitatively

[35] In this model, all variation in wealth is related to prices and it is transitory, because the quantities of the asset are fixed.

consistent with empirical observation, and it stands in sharp contrast to the results of the standard paradigm --the infinite-lived representative consumer model-- in which return and price dynamics closely mirror the dynamics of the dividends, and prices and expected returns are not autocorrelated when the fundamentals are not. The 3-generation OLG model provides a simple and intuitive explanation of the long-run negative autocorrelation of returns. A byproduct of this negative autocorrelation is that the correlation between aggregate consumption and wealth is low.

In our model, the *young* receive a consumption endowment while the *middle-aged* receive assets that pay risky dividends in the following period. We assume away market frictions, incomplete markets, asymmetric information and all types of irrationality.

We show that negative autocorrelation in expected returns, risk premia, and prices (and the resulting "excess" and time-varying volatility) is a fundamental property of the model, because consumers' excess supply for assets is price-elastic. The *middle-aged* can choose how much of their asset endowment to sell, because they have an alternative use: to hold the assets over to the next period and consume the dividends. The excess supply of the *middle-aged* depends on the quantity of assets they purchase when young, which in turns depends on the prices they encountered at that time. In this way, the history of each generation affects the date=t equilibrium, even though the distributions of endowments and dividends do not change over time.

Price-elastic excess supply of assets is a feature of everyday markets. It is perhaps surprising that including this feature in an OLG model generates return and asset price properties at such great variance with our usual concept of efficient market equilibrium. Yet, the properties of our model are qualitatively similar to those in the data. We conclude that the observed predictability of returns are difficult to explain because our ideas of market efficiency are informed mainly by the properties of representative consumer models, which do not capture the effects of intergenerational trading.

Assuming a 3-period lifespan necessarily confines the analysis to long-run behavior. To address short-run properties of asset returns, the number of periods consumers live must be increased. The interaction of many generations each with a slightly different financial history will undoubtedly produce much richer dynamics. Such an extension presents a serious technical challenge and is the subject of ongoing research.

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APPENDIX 1: THE INDIVIDUAL CONSUMER'S OPTIMIZATION PROBLEM

A state-by-state rewriting of the individual maximization problem in Langrangian form yields (s is the state of birth),

(A1.1)

$$\underset{\left\{\begin{matrix} C_1(s) \\ G_1(s), B_1(s) \\ C_2(sl), G_2(sl), \\ B_2(sl) C_3(slm) \end{matrix}\right\}}{\text{Max}} E_t\{Z(1, sh_{+1}h_{+2})\} = U[C_1(s)] + d\sum_{l=g,b} p_{sl} U[C_2(sl)] + d^2 \sum_{l=g,b} \sum_{m=g,b} p_{lm} U[C_3(slm)]$$

(A1.2a) $\quad - l1(s)\langle C_1(s) + p_G(h_{-1}s)G_1(s) + p_B(h_{-1}s)B_1(s) - E(s)\rangle,$

(A1.2b) $\quad - l2(sl)\{C_2(sl) - d_G(l)G_1(s) - d_B(l)B_1(s) - p_G(l)[K - G_2(sl)] - p_B(l)[K - B_2(sl)]\},$

(A1.2c) $\quad - l3(slm)\{C_3(slm) - d_G(m)G_2(sl) - d_B(m)B_2(sl)\};\ s,l,m = g,b.$

First Order Conditions

The FOCs from differentiating w.r.t. *young*, *middle-aged* and *old* consumptions yield,

$$U'[C_1(s)] = l1(s), \qquad p_{sg} d^2 \{p_{gg} U'[C_3(sgg)]\} = l3(sgg),$$
$$p_{sg} dU'[C_2(sg)] = l2(sg), \qquad p_{sg} d^2 \{p_{gb} U'[C_3(sgb)]\} = l3(sgb),$$
$$p_{sb} dU'[C_2(sb)] = l2(sb), \qquad p_{sb} d^2 \{p_{bg} U'[C_3(sbg)]\} = l3(sbg),$$
$$p_{sb} d^2 \{p_{bb} U'[C_3(sbb)]\} = l3(sbb).$$

where,

(A1.3a,b) $\quad q_{hg} = \dfrac{l2(sg)}{l1(s)} = dp_{sg}\left\{\dfrac{U'[C_2(sg)]}{U'[C_1(s)]}\right\}, \qquad q_{hb} = \dfrac{l2(sb)}{l1(s)} = dp_{sb}\left\{\dfrac{U'[C_2(sb)]}{U'[C_1(s)]}\right\},$

(A1.3c,d) $\quad q_{hgg} = \dfrac{l3(sgg)}{l2(sg)} = dp_{gg}\left\{\dfrac{U'[C_3(sgg)]}{U'[C_2(sg)]}\right\}, \qquad q_{hgb} = \dfrac{l3(sgb)}{l2(sg)} = dp_{gb}\left\{\dfrac{U'[C_3(sgb)]}{U'[C_2(sg)]}\right\},$

(A1.3e,f) $\quad q_{hbg} = \dfrac{l3(sgb)}{l2(sb)} = dp_{bg}\left\{\dfrac{U'[C_3(sbg)]}{U'[C_2(sb)]}\right\}, \qquad q_{hbb} = \dfrac{l3(sbb)}{l2(sb)} = dp_{bb}\left\{\dfrac{U'[C_3(sbb)]}{U'[C_2(sb)]}\right\},$

are the individual's probability-weighted discounted one-period marginal rates of substitution, which the individual equates to the economy-wide state prices: $q_{hs_{+1}}$ in youth and $q_{hs_{+1}s_{+2}}$ in middle age. Differentiating with respect to share holdings of each asset to the *young* $k(1,h)$, $k=\{a,b\}$, yields,

(A1.4) $\quad p_k(h) = \dfrac{l2(sg)}{l1(s)} d_k(g) + \dfrac{l2(sb)}{l1(s)} d_k(b).$

Similarly, differentiating with respect to the share holdings of the *middle-aged k2*(*sg*) and *k2*(*sb*) yields:

(A1.5a) $$p_k(hg) = \frac{l3(sgg)}{l2(sg)} d_k(g) + \frac{l3(sgb)}{l2(sg)} d_k(b),$$

(A1.5b) $$p_k(hb) = \frac{l3(sbg)}{l2(sb)} d_k(g) + \frac{l3(sbb)}{l2(sb)} d_k(b).$$

Since each person equalizes her personal marginal rate of substitution to the state prices, these equations become pricing equations:

(A1.6)
$$\begin{aligned} p_k(h) &= q_{hg} d_k(g) + q_{hb} d_k(b), \\ p_k(hg) &= q_{hgg} d_k(g) + q_{hgb} d_k(b), \\ p_k(hb) &= q_{hbg} d_k(g) + q_{hbb} d_k(b). \end{aligned}$$

These equations merely show that share prices are the present value of the dividends. Finally, differentiating with respect to the Langrangian multipliers we get back the 7 budget constraints.

(A1.7a) $C_1(s) = E(s) - p_G(h)G_1(s) - p_B(h)B_1(s)$,

(A1.7b) $C_2(sg) = d_G(g)G_1(s) + p_G(hg)[K - G_2(sg)] + p_B(hg)[K - B_2(sg)]$,

(A1.7c) $C_2(sb) = d_B(b)B_1(s) + p_G(hb)[K - G_2(sb)] + p_B(hb)[K - B_2(sb)]$,

(A1.7d,e) $C_3(sgg) = d_G(g)G_2(sg)$, $C_3(sgb) = d_B(b)B_2(sg)$,

(A1.7f,g) $C_3(sbg) = d_G(g)G_2(sb)$, $C_3(sbb) = d_B(b)B_2(sb)$.

A a "*young*" consumer takes as given the 14 asset prices that she may encounter over her lifetime and solves her optimal lifetime consumption and asset holding plan. Seven consumptions –$C_1(s)$, $C_2(sg)$, $C_2(sb)$, $C_3(sgg)$, $C_3(sgb)$, $C_3(sbg)$, $C_3(sbb)$-- and six asset holdings –$G_1(s)$, $G_2(sg)$, $G_2(sb)$, $B_1(s)$, $B_2(sg)$, $B_2(sb)$. That gives 13 variables to solve for, and there are 6 MRS equations and 7 budget constraints. In addition, we get 6 pricing equations.

Another way to summarize the budget constraints is to express them in terms of one intertemporal budget constraint:

(A1.8)

$$\begin{aligned} &E_s + q_{hg}[p_G(hg) + p_B(hg)]K + q_{hb}[p_G(hb) + p_B(hb)]K \\ &\quad = C_1(s) + q_{hg}C_2(sg) + q_{hb}C_2(sb) + q_{hg}[q_{hgg}C_3(sgg) + q_{hgb}C_3(sgb)] + q_{hb}[q_{hbg}C_3(sbg) + q_{hbb}C_3(sbb)] \end{aligned}$$

APPENDIX 2: SOLUTIONS FOR THE BENCHMARK MODELS

Appendix 2A: The ILRC (Representative Consumer) Model

The equilibrium solution is immediate in this case, because the representative consumer consumes all that is available in each period, and thus has no meaningful decision to make. The state prices reflect total available endowment, and the asset prices are shadow prices that reflect the equality of latent excess demand and supply.

The equilibrium consumption is given by, $C(s) = E_s + Kd_s$. This is enough to compute the state prices and the shadow prices of the assets.

(A2.1a,b) $$q_{gg} = dp_{gg}\left(\frac{E_g + Kd_G}{E_g + Kd_G}\right)^{g} = dp_g, \quad q_{gb} = dp_{gb}\left(\frac{E_g + Kd_G}{E_b + Kd_B}\right)^{g} = dp_b ?^{g},$$

(A2.1c,d) $$q_{bb} = dp_{bb}\left(\frac{E_b + Kd_B}{E_b + Kd_B}\right)^{g} = dp_b, \quad q_{bg} = dp_{bg}\left(\frac{E_b + Kd_B}{E_g + Kd_G}\right)^{g} = dp_g ?^{-g},$$

where, $? = \dfrac{E_g + Kd_g}{E_b + Kd_b} > 1$, and probabilities are state-independent.

Asset prices then are:

(A2.2a) $$P_G(g) = q_{gg} d_G = dp_g d_G, \quad P_G(b) = q_{bg} d_G = dp_g ?^{-g} d_G,$$

(A2.2b) $$P_B(b) = q_{bb} d_B = dp_b d_B, \quad P_B(g) = q_{gb} d_B = dp_b ?^{g} d_B.$$

Appendix 2B: The 2-OLG Competitive Model

The appropriate transformation of the 3-period OLG model to a 2-period one is as follows:

All parameters stay the same. The young still receive $E(s)$ endowments and the old (rather than the middle-aged) receive K units of each of the two risky 1-period assets. Since these assets do not produce dividends until the following period, the old must sell them to their only costumer, the young. The young buy these assets with some of their endowment income.

The old consume both the dividends of the assets they bring into the period and the proceeds from the sale of their endowment of assets.

This model is strictly recursive, i.e., past history doesn't matter, so that there is a single solution for a *good* state and another one for a *bad* state. The reason follows.

For simplicity, assume that $p_{sg} = p_g$, $p_{sb} = p_b$. Consider the *good* (g) states at dates t and $t+n$. At both dates the supply of the consumption good is $E(g) + Kd_G$.

At both dates, the young are endowed with $E(g)$, the old are endowed with K, and the young purchase *all* the assets of the old. Thus, the past has no impact on the demand and supply of the present state. The solution at both dates must be the same. The same argument applies identically to *bad* states.[36]

Thus, consumption is not indexed by date or history.

We solve the problem by first solving the individual consumer's maximization problem. Then we impose the asset market clearing condition. Consider the individual's lifetime maximization problem:

$$\text{(A2.3)} \qquad \underset{\left\{\substack{C_1(s), C_2(s_{+1}) \\ G_1(s), B_2(s)}\right\}}{\text{Max}} \; \mathrm{E}_t\{Z(s, s_{+1})\} = \; U[C_1(s)] + \mathrm{E}_t\{dU[C_2(s_{+1})]\},$$

.

subject to the age=1, 2, budget constraints, respectively:

$$\text{(A2.4a)} \qquad C_1(s) = E(s) - P_G(s)G_1(s) - P_B(s)B_1(s),$$

$$\text{(A2.4b)} \qquad C_2(s_{+1}) = G_1(s)d_G(s_{+1}) + B_1(s)d_B(s_{+1}) + P_G(s_{+1})K + P_B(s_{+1})K.$$

The derivation of the first order conditions follows. At any date=t, the *young* and *old* jointly determine equilibrium allocations. The *young* equate their expected marginal rates of substitution to the market's, and the *old* do not have a relevant marginal rate of substitution.

$$\text{(A2.5a)} \qquad \underset{\left\{\substack{C_1(s), G_1(s), B_1(s) \\ C_2(sg), C_2(sb)}\right\}}{Max} \; E_t\{W(s, s_{+1})\} = U[C_1(s)] + d\{p_g U[C_2(g)] + p_b U[C_2(b)]\},$$

$$\text{(A2.5b)} \qquad - l1(s)\{E(s) - C_1(s) - P_G(s)G_1(s) - P_B(s)B_1(s)\},$$

$$\text{(A2.5c)} \qquad - l2(g)\{C_2(g) - G_1(s)d_G(g) - P_G(g)K - P_B(g)K\},$$

$$\text{(A2.5d)} \qquad - l2(b)\{C_2(b) - B_1(s)d_B(b) - P_G(b)K - P_B(b)K\}.$$

Since $d_G(b) = d_B(g) = 0$, the associated terms have been deleted from the equations above.

First Order Conditions:
The FOCs from differentiating w.r.t. *young*, *middle-aged* and *old* consumptions yield,

[36] If the transition probabilities depend on the immediate predecessor state, then there will be four rather than two equilibria, one each for states *gg*, *gb*, *bg*, and *bb*.

(A2.6) $C_1(s)^{-g} = l1(s),\ p_g d C_2(g)^{-g} = l2(g),\ p_b d C_2(b)^{-g} = l2(b).$

Where the *young* consumers' probability-weighted discounted one-period marginal rates of substitution is equated to the economy-wide state prices: q_{sg}, q_{sg},

(A2.7a,b) $$q_{sg} = \frac{l2(g)}{l1(s)} = dp_g \left[\frac{C_1(s)}{C_2(g)}\right]^g, \qquad q_{sb} = \frac{l2(b)}{l1(s)} = dp_b \left[\frac{C_1(s)}{C_2(b)}\right]^g,$$

Differentiating with respect to share holdings of each asset to the *young* $k1(h)$, $k=\{a,b\}$, yields the general relation,

(A2.8) $$P_k(s) = \frac{l2(g)}{l1(s)} d_k(g) + \frac{l2(b)}{l1(s)} d_k(b).$$

Since each person equalizes her personal marginal rate of substitution to the state prices, these equations become pricing equations:

(A2.9) $$P_k(s) = q_{sg} d_k(g) + q_{sb} d_k(b).$$

Under our payoff structure this yields:

(A2.10a,b) $P_G(g) = q_{gg} d_G(g),\ \ P_G(b) = q_{bg} d_G(g),$

(A2.10c,d) $P_B(g) = q_{gb} d_B(b),\ \ P_B(b) = q_{bb} d_B(b).$

The asset market clearing condition is that at the end of date=t all shares outstanding are held by the *young*,

(A2.11) $$G_1(s) = B_1(s) = K.$$

The simplest approach is to impose the asset market clearing condition directly to equations (A2.1). This gives,

(A2.12a) $C_1(g) = E(g) - P_G(g)K - P_B(g)K,$

(A2.12b) $C_1(b) = E(b) - P_G(b)K - P_B(b)K,$

(A2.12c) $C_2(g) = Kd_G + P_G(g)K + P_B(g)K,$

(A2.12d) $C_2(b) = Kd_B + P_G(b)K + P_B(b)K.$

Substitute out the asset prices to get,

(A2.13a) $$C_1(g) = E(g) - K\lfloor q_{gg} d_G + q_{gb} d_B \rfloor,$$

(A2.13b) $$C_1(b) = E(b) - K\lfloor q_{bg} d_G + q_{bb} d_B \rfloor,$$

(A2.13c) $$C_2(g) = Kd_G + K\lfloor q_{gg} d_G + q_{gb} d_B \rfloor,$$

(A2.13d) $$C_2(b) = Kd_B + K\lfloor q_{bg} d_G + q_{bb} d_B \rfloor.$$

Now substitute the consumptions into the state prices to get,

(A2.14a) $$C_1(g) = E(g) - K\left[F_g\left(\frac{C_1(g)}{C_2(g)}\right)^g d_G + F_b\left(\frac{C_1(g)}{C_2(b)}\right)^g d_B\right],$$

(A2.14b) $$C_1(b) = E(b) - K\left[F_g\left(\frac{C_1(b)}{C_2(g)}\right)^g d_G + F_b\left(\frac{C_1(b)}{C_2(b)}\right)^g d_B\right],$$

(A2.14c) $$C_2(g) = Kd_G + K\left[F_g\left(\frac{C_1(g)}{C_2(g)}\right)^g d_G + F_b\left(\frac{C_1(g)}{C_2(b)}\right)^g d_B\right],$$

(A2.14d) $$C_2(b) = Kd_B + K\left[F_g\left(\frac{C_1(b)}{C_2(g)}\right)^g d_G + F_b\left(\frac{C_1(b)}{C_2(b)}\right)^g d_B\right],$$

where $F_g \equiv dp_g$; $F_b \equiv dp_b$ are constants.

These equations have to be solved simultaneously.

As in the 3-OLG case, the individual's budget constraint cannot be reduced to the economy-wide resource constraint used by the social planner. Thus the competitive solution, though stationary, is not the SP solution.

Appendix 2C: The 3-OLG Social Planner Model

The social planner (SP) solves the following maximization problem that includes arbitrary weights across individuals and generations.

(A2.15) $$Max \quad \sum_t \left\{ \sum_h \Theta_{1,t} p_{h-1,h} U[C_1(h)] + \Theta_{2,t} p_{h-1,h} dU[C_2(h)] + \Theta_{3,t} p_{h-1,h} d^2 U[C_3(h)] \right\}$$
$$\text{s.t.,} \quad C_1(s) + C_2(s) + C_3(s) = E(s) + K\sum_k d_k(s),$$

There is no economic connection between successive time periods, because consumption allocations at date=*t* cannot affect consumption allocations at date=*t+1* or beyond. Thus, the

SP has to make only 2 allocations, for s = *g*, *b*. To simplify the algebra, we also assume that, $\Theta_{G,t} = 1; \quad \forall\, G,t.$

Normalize the problem and introduce some notation simplifications,

(A2.16)
$$Max \quad p_{s-1,s}U[C_1(s)] + p_{s-1,s}dU[C_2(s)] + p_{s-1,s}d^2U[C_3(s)],$$
$$\text{s.t.,} \quad C_1(s) + C_2(s) + C_3(s) = E(s) + Kd_k(s), \text{ for } s=g,b.$$

We drop the summation sign for the dividends, because we assume all along an orthogonal payout scheme.

The first order conditions, in addition to the resource constraint are as follows:

(A2.17)
$$p_{s-1,s}U'[C_1(s)] - l = 0, \quad p_{s-1,s}U'[C_2(s)] - l = 0, \quad p_{s-1,s}U'[C_3(s)] - l = 0.$$

These conditions simplify to the following:

(A2.18)
$$p_{s-1,s}U'[C_1(s)] = p_{s-1,s}dU'[C_2(s)] = p_{s-1,s}d^2U'[C_3(s)].$$

Substituting in the RRA utility function and canceling out the probabilities we get, $C_1^{-g} = dC_2^{-g} = d^2C_3^{-g}$, which simplifies to, $C_1 = d^{-1/g}C_2 = d^{-2/g}C_3$, dropping the index *s*, with the understanding that there are two solutions (*s*=*g*, *b*).

The rule that the SP divides available consumption is given by,

(A2.19a)
$$C_1\left(1 + d^{1/g} + d^{2/g}\right) = E + Kd_k, \quad C_1 = \frac{E + Kd_k}{1 + d^{1/g} + d^{2/g}},$$

(A2.19b)
$$C_3 = C_2 d^{1/g}, \quad C_2 = C_1 d^{1/g}.$$

There are 4 "shadow" state prices in this economy, and they are as follows:

(A2.20a,b)
$$q_{gg} = p_{gg}d\left(\frac{C_2(g)}{C_1(g)}\right)^{-g} = p_{gg}d\left(\frac{C_3(g)}{C_2(g)}\right)^{-g}, \; q_{gb} = p_{gb}d\left(\frac{C_2(b)}{C_1(g)}\right)^{-g} = p_{gb}d\left(\frac{C_3(b)}{C_2(g)}\right)^{-g},$$

(A2.20c,d)
$$q_{bg} = p_{bg}d\left(\frac{C_2(g)}{C_1(b)}\right)^{-g} = p_{bg}d\left(\frac{C_3(g)}{C_2(b)}\right)^{-g}, \quad q_{bb} = p_{bb}d\left(\frac{C_2(b)}{C_1(b)}\right)^{-g} = p_{bb}d\left(\frac{C_3(b)}{C_2(b)}\right)^{-g}.$$

The consumption ratios are,

For the g to g, and b to b transitions, there is one ratio, $\frac{C_2(g)}{C_1(g)} = \frac{C_3(g)}{C_2(g)} = \frac{C_2(b)}{C_1(b)} = \frac{C_3(b)}{C_2(b)} = d^{1/g}$, while for the g to b and b to g transitions there is another. Note that,

(A2.21) $$\frac{C_1(g)}{C_1(b)} = \frac{E_g + Kd_g}{E_b + Kd_b} = ? > 1.$$

Therefore,

(A2.22) $$\frac{C_2(g)}{C_1(b)} = ?\, d^{1/g}, \text{ and } \frac{C_2(b)}{C_1(g)} = ?^{-1} d^{1/g}.$$

Substitute these relations to solve for the state prices:

$q_{gg} = p_{gg}$, $q_{bb} = p_{bb}$, $q_{gb} = p_{gb}?^{g}$, $q_{bg} = p_{bg}?^{-g}$. Asset prices then are:

(A2.23a,b) $P_G(g) = q_{gg} d_G = p_{gg} d_G$, $P_G(b) = q_{bg} d_G = p_{bg}?^{-g} d_G$,

(A2.23c,d) $P_B(b) = q_{bb} d_B = p_{bb} d_B$, $P_B(g) = q_{gb} d_B = p_{gb}?^{g} d_B$.

APPENDIX 3: PROOF THAT THE SOCIAL PLANNER SOLUTION IS NOT FEASIBLE IN A COMPETITIVE EQUILIBRIUM.

Theorem 1: The social planner solution is not attainable in a competitive market.

Proof: The social planner solves the equilibrium problem constrained by available resources (equation 12):

(12) $$C_1(s)+C_2(s)+C_3(s)=E(s)+K\sum_k d_k(s).$$

It produces a recursive solution in which the consumption of all 3 generations depends only on the current state.

In contrast, the competitive equilibrium has an additional constraint, the wealth constraint of each individual consumer. This can be expressed as:

(A3.1) $$\begin{aligned} E(s)+\sum_{k=G,B}\sum_{s_{+1}=g,b} q_{s,s_{+1}} P_k(s,s_{+1})K &= C_1(s)+\sum_{g,b} q_{s,s_{+1}} C_2(s,s_{+1}) \\ &+ \sum_{s_{+1}=g,b}\sum_{s_{+2}=g,b} q_{s,s_{+1},s_{+2}} q_{s,s_{+1}} C_3(s,s_{+1},s_{+2}). \end{aligned}$$

It is clear by inspection that these two constraints are not equal, since (A3.1) contains future state-dependent consumptions that are not relevant to the social planner. Thus, the social planner solution will not satisfy (A3.1)

There are two present value constraints, for states g and b. We substitute into the constraint the social planner solution and ask whether the present value constraint is identical to the resource constraint. The constraints for the two possible states are:

(A3.2.a) $$\begin{aligned} E(g)+q_{gg}[P_G(g)+P_B(g)]K+q_{gb}[P_G(b)+P_B(b)]K &= C_1(g)+q_{gg}C_2(g)+q_{gb}C_2(b) \\ &+ q_{gg}\left[q_{gg}C_3(g)+q_{gb}C_3(b)\right]+q_{gb}\left[q_{bg}C_3(g)+q_{bb}C_3(b)\right]. \end{aligned}$$

(A3.2.b) $$\begin{aligned} E(b)+q_{bg}[P_G(g)+P_B(g)]K+q_{bb}[P_G(b)+P_B(b)]K &= C_1(b)+q_{bg}C_2(g)+q_{bb}C_2(b) \\ &+ q_{bg}\left[q_{gg}C_3(g)+q_{gb}C_3(b)\right]+q_{bb}\left[q_{bg}C_3(g)+q_{bb}C_3(b)\right]. \end{aligned}$$

The state prices are from equations (6) and the asset prices are from equations (11).

(17) $$q_{gg}=p_g,\quad q_{bb}=p_b,\quad q_{gb}=p_bF^{-g},\quad q_{bg}=p_gF^{g}.$$

(A3.3) $P_G(g) = q_{gg} d_G$, which reduces to, $P_G(g) = p_{gg} d_G$,
$P_G(b) = q_{bg} d_G$, which reduces to, $P_G(b) = p_{bg} F^g d_G$,
$P_B(b) = q_{bb} d_B$, which reduces to, $P_B(b) = p_{bb} d_B$,
$P_B(g) = q_{gb} d_B$, which reduces to, $P_B(g) = p_{gb} F^{-g} d_B$.

Substitute all this into equations (A3.2a&b) and collect terms to get,

(A3.4a)
$$\begin{aligned} &E(g) + p_g \left[p_g d_G + p_b F^{-g} d_B\right] K + p_b F^{-g} \left[p_g F^g d_G + p_b d_B\right] K \\ &\quad = C_1(g) + p_g C_2(g) + p_b F^{-g} C_2(b) \\ &\qquad + p_g \left[p_g C_3(g) + p_b F^{-g} C_3(b)\right] + p_b F^{-g} \left[p_g F^g C_3(g) + p_b C_3(b)\right]. \end{aligned}$$

(A3.4b)
$$\begin{aligned} &E(b) + p_g F^g \left[p_g d_G + p_b F^{-g} d_B\right] K + p_b \left[p_g F^g d_G + p_b d_B\right] K \\ &\quad = C_1(b) + p_g F^g C_2(g) + p_b C_2(b) \\ &\qquad + p_g F^g \left[p_g C_3(g) + p_b F^{-g} C_3(b)\right] + p_b \left[p_g F^g C_3(g) + p_b C_3(b)\right]. \end{aligned}$$

collect terms further to get:

(A3.5a)
$$\begin{aligned} &E(g) + K\left\{\left(p_g^2 + p_g p_b\right) d_G + \left(p_g p_b F^{-g} + p_b^2 F^{-g}\right) d_B\right\} \\ &\quad = C_1(g) + p_g C_2(g) + p_b F^{-g} C_2(b) + p_g C_3(g) + p_b F^{-g} C_3(b). \end{aligned}$$

(A3.5b)
$$\begin{aligned} &E(b) + K\left\{\left(p_g^2 F^g + p_g p_b F^g\right) d_G + \left(p_g p_b + p_b^2\right) d_B\right\} \\ &\quad = C_1(b) + p_g F^g C_2(g) + p_b C_2(b) + p_g F^g C_3(g) + p_b C_3(b). \end{aligned}$$

(A3.6a)
$$\begin{aligned} &E(g) + K\left\{p_g d_G + p_b F^{-g} d_B\right\} = \\ &\quad C_1(g) + p_g C_1(g) d^{1/g} + p_b F^{-g} C_1(b) d^{1/g} + p_g C_1(g) d^{2/g} + p_b F^{-g} C_1(b) d^{2/g}. \end{aligned}$$

(A3.6b)
$$\begin{aligned} &E(b) + K\left\{p_g F^g d_G + p_b d_B\right\} = \\ &\quad C_1(b) + p_g F^g C_1(g) d^{1/g} + p_b C_1(b) d^{1/g} + p_g F^g C_1(g) d^{2/g} + p_b C_1(b) d^{2/g}. \end{aligned}$$

(A3.7a)
$$\begin{aligned} &E(g) + K\left\{p_g d_G + p_b F^{-g} d_B\right\} = \\ &\quad \left\{1 + p_g d^{1/g} + p_g d^{2/g}\right\} C_1(g) + p_b F^{-g} d^{1/g} \left\{1 + d^{1/g}\right\} C_1(b). \end{aligned}$$

(A3.7b)
$$E(b)+K\{p_g F^g d_G + p_b d_B\} = \left\{1+p_b d^{1/g} + p_b d^{2/g}\right\} C_1(b) + p_g F^g d^{1/g}\left\{1+d^{1/g}\right\} C_1(g).$$

(A3.8a)

$$E(g)+K\{p_g d_G + p_b F^{-g} d_B\} = \Gamma\left\{1+p_g d^{1/g} + p_g d^{2/g}\right\}(E(g)+Kd_G) + p_b F^{-g} d^{1/g}\Gamma\left\{1+d^{1/g}\right\}(E(b)+Kd_B).$$

(A3.8b)
$$E(b)+K\{p_g F^g d_G + p_b d_B\} = \Gamma\left\{1+p_b d^{1/g} + p_b d^{2/g}\right\}(E(b)+Kd_B) + p_g F^g d^{1/g}\Gamma\left\{1+d^{1/g}\right\}(E(g)+Kd_G).$$

where, $\Gamma \equiv \left(1+d^{1/g}+d^{2/g}\right)^{-1}$.

Now write the resource constraint for state $s = g, b$, from equation (8 of the text) above,

(A3.9) $$E(s)+Kd_s = C_1(s)+C_2(s)+C_3(s).$$

This becomes,

(A3.10) $$E(s)+Kd_s = C_1(s)+C_1(s)d^{1/g} + C_1(s)d^{2/g},$$

(A3.11) $$E(s)+Kd_s = C_1(s)\Gamma^{-1}.$$

It is clear that the present value constraints (equations A3.8a & b) are not identical to the resource constraint (equation A3.11). In particular, in order for the two constraints to coincide, the following conditions must be met.

(A3.12.a) $$p_g d_G + p_b F^{-g} d_B = d_B,$$

(A3.12.b) $$p_g F^g d_G + p_b d_B = d_B,$$

(A3.13) $$p_s\left(1+d^{1/g}\right) = 1+d^{1/g},$$

(A3.14) $$C_1(s)p_s F^g d^{1/g}\left\{1+d^{1/g}\right\}=0,$$

These conditions are fulfilled only when there is no uncertainty, i.e., p_g = 1.0, p_b = 0.0, and by definition F = 1; in this case only (A3.12.a), (A3.13), and (A3.14) apply. Therefore, it is clear that the social planner solution is not attainable, in general, in a competitive market.

An implication of the above result is that there is no other competitive market solution of this class, i.e., consumption outcomes that depend *only* on the current state of the world. This is because the competitive market solution has to obey the individuals' wealth constraint, in addition to the other 1st order conditions that the S.P. solution obeys. As we show above these wealth constraints are not redundant, thus the model is overidentified for this class of solutions and there is no solution.

Q.E.D.

APPENDIX 4: PROOF THAT THE COMPETITIVE SOLUTION IS APERIODIC

Theorem 2: The competitive solution in this economy is path-dependent, even though all good and bad states have path-independent resource allocations.

Proof:

Start with the following 3-period periodic equilibrium. $C_1(g)$, $C_1(b)$, $C_2(gg)$, $C_2(gb)$, $C_2(bg)$, $C_2(bb)$, $C_3(ggg)$, $C_3(ggb)$, $C_3(gbg)$, $C_3(gbb)$, $C_3(bgg)$, $C_3(bgb)$, $C_3(bbg)$, $C_3(gbb)$. The equilibrium is one in which each of the 8 possible life paths has its own consumption experience, and that past history is not relevant. We show that this equilibrium collapses to the social planner equilibrium, which we have already shown is not feasible in the competitive economy.

The equalization of the marginal rates of substitution implies the following equalities:

(A4.1a) $$\frac{C_1(g)}{C_2(gg)} = \frac{C_2(gg)}{C_3(ggg)},$$ (A4.1b) $$\frac{C_1(g)}{C_2(gb)} = \frac{C_2(gg)}{C_3(ggb)},$$

(A4.1c) $$\frac{C_1(b)}{C_2(bg)} = \frac{C_2(gb)}{C_3(gbg)},$$ (A4.1d) $$\frac{C_1(b)}{C_2(bb)} = \frac{C_2(gb)}{C_3(gbb)},$$

(A4.1e) $$\frac{C_1(g)}{C_2(gg)} = \frac{C_2(bg)}{C_3(bgg)},$$ (A4.1f) $$\frac{C_1(g)}{C_2(gb)} = \frac{C_2(bg)}{C_3(bgb)},$$

(A4.1g) $$\frac{C_1(b)}{C_2(bg)} = \frac{C_2(bb)}{C_3(bbg)},$$ (A4.1h) $$\frac{C_1(b)}{C_2(bb)} = \frac{C_2(bb)}{C_3(bbb)}.$$

These equalities are derived as follows: imagine being in state g whose parent state is also g. The *young* equalize their marginal rates of substitution (MRS) between their current and future consumptions, e.g., $C_1(g)$, and $C_2(gg)$ and $C_2(gb)$, while the *middle-aged* equalize to the same MRS $C_2(gg)$, and $C_3(ggg)$ and $C_3(ggb)$. Do this until all the possible combinations are exhausted.

The economy-wide resource constraints for the g and b states are,

(A4.2a) $$Kd_G = C_1(g) + C_2(gg) + C_3(ggg),$$

(A4.2b) $$Kd_G = C_1(g) + C_2(gg) + C_3(bgg),$$

(A4.2c) $$Kd_G = C_1(g) + C_2(bg) + C_3(gbg),$$

(A4.2d) $$Kd_G = C_1(g) + C_2(bg) + C_3(bbg),$$

(A4.2e) $$Kd_B = C_1(b) + C_2(gb) + C_3(ggb),$$

(A4.2f) $$Kd_B = C_1(b) + C_2(gb) + C_3(bgb),$$

(A4.2g) $$Kd_B = C_1(b) + C_2(bb) + C_3(gbb),$$

(A4.2h) $$Kd_B = C_1(b) + C_2(bb) + C_3(bbb).$$

Equations (A4.2) imply the following equalities:

(A4.3) $$C_3(ggg) = C_3(bgg) \equiv C_3(gg), \quad C_3(gbg) = C_3(bbg) \equiv C_3(bg),$$
$$C_3(ggb) = C_3(bgb) \equiv C_3(gb), \quad C_3(gbb) = C_3(bbb) \equiv C_3(bb).$$

These equalities substituted into the corresponding equations (A4.1) provide the following additional equalities:

(A4.4) $$C_2(gg) = C_2(bg) \equiv C_2(g), \; C_2(gb) = C_2(bb) \equiv C_2(b).$$

Going back once more to the resource constraints (equations A.5.2) we see that,

(A4.5) $$C_3(gg) = C_3(bg) \equiv C_3(g), \; C_3(gb) = C_3(bb) \equiv C_3(b).$$

This proves that this particular periodicity collapses to an equilibrium that depends only on the current state of the economy. Q.E.D.

The obvious question is: what if we expand the recursion to 4 periods or more. In order to do that we must index consumptions by the parent state (or states). Write the consumption of a *young* consumer born in state=*g*, with a parent state=*g* as $C_1(g;g)$.

It is important to realize that this implies $C_2(gg;g)$, $C_2(gb;g)$, $C_3(ggg;g)$, $C_3(ggb;g)$, $C_3(gbg;g)$, $C_3(gbb;g)$. In other words, for such a recursion to make sense, it must be that *all* the subsequent consumptions also have to be indexed by the same pre-birth parent state.

It is clear from this observation that the above proof carries through for any periodicity. This is because the number of equations (MRS equalities and resource constraints) will expand proportionally to the number of states consumptions are indexed by. For example, a 4-period periodicity implies twice as many equations as a 3-period periodicity etc. Thus, the same method of searching for *old-aged* consumption equalities, substituting them back into the resource constraints and repeating the cycle applies, and produces the same result. Thus, periodicity of any finite length collapse to an equilibrium that depends only on the current state.

By Theorem 1 a solution that depends only on the current state of the economy cannot be a competitive market equilibrium. Thus, the competitive market equilibrium cannot be periodic.

APPENDIX 5: DERIVATION OF THE DIFFERENCE EQUATIONS:

The solution reduces the problem to solving for the shares held by successive *middle-aged* cohorts into each date=t along all possible state histories, h_T. We develop two sets of equations. The first is a recursive relation for $R(h_{-1})$ that incorporates the first order conditions over successive states. The second is an implicit recursive relation for asset holdings that the middle age bring into the state, $G_1(h_{-1})$ or $B_1(h_{-1})$, which are compatible with first order and the asset market clearing conditions.

Numbering Convention:
To develop the equations we need a numbering convention that can reference every state on the tree. On a tree of length T there are 2^{T+1}-1 states. There is also a special initial state before the start of the world at date=0. The individual states are numbered consecutively, starting from the top of the tree each period. There are 2^t states at each date. The initial state is state i=0, the state at t=0 is i=1, the states at t=1 are i=2-3, states at t=2 are i=4-7, and so fourth. From date=0 to date=T including the initial state, the current state s is indexed by i= 0,1,…, 2^{T+1}-1. By convention, an odd state is a "bad" state and an even state is a "good" state. The successor states of a particular parent state, $s=i$, are states s=2i (sg) and s=2i+1 (sb). Similarly, the parent state is Integer(i/2).

The Difference Equations:
We derive first the recursive relation for $R(i)$. We will only illustrate the derivation for transitions into good states, noting that the expression for bad states has an identical form. From a particular parent i write the definition for $R(i)$,

(A5.1a) $$R(i) \equiv \frac{C_2(i)}{C_1(i)} = R^g(i) \equiv \frac{C_3(2i)}{C_2(2i)} = R^b(i) \equiv \frac{C_3(2i+1)}{C_2(2i+1)},$$

where there is a good (2i) and a bad successor state (2i+1) next period. The ratio that refers to the future *good* states, $R^g(i)$, can be re-expressed using the definition for $R(2i)$,

(A5.2) $$R^g(i) = \frac{C_3(2i)}{C_2(2i)} = \frac{C_3(2i)}{C_1(2i)R(2i)}.$$

Further substitute in the *old*'s budget (*old* at 2i) constraint expressed in terms of shares the *middle-aged* brings into *2i* to get,

(A5.3) $$R^g(i) = \frac{d_G(g)[K - G_1(i)]}{C_1(2i)R(2i)}.$$

It is useful to express $C_1(2i)$ as a function of $R(2i)$, and $G_1(i)$. Start with the *middle-aged* budget constraint,

(A5.4) $$C_2(2i) = d_G(g)G_1(i) + P_G(2i)[K - G_2(2i)] + P_B(2i)[K - B_2(2i)],$$

noting that the bad state asset does not pay dividends in a good state and is therefore not in the equation. In equilibrium the proceeds from the sales of shares to the *young* are part of the *young*'s endowment not consumed, adjusted for the size of the generation,

(A5.4) $$\begin{aligned} P_G(2i)[K - G_2(2i)] + P_B(2i)[K - B_2(2i)] &= (P_G(2i)[G_1(2i)] + P_B(2i)[B_1(2i)]), \\ &= [E(g) - C_1(2i)]. \end{aligned}$$

Substituting this into the *middle-aged* budget constraint yields,

(A5.5) $$C_2(2i) = d_G(g)G_1(i) + [E(g) - C_1(2i)].$$

Substitute the definition for $R(2i)$ and solve for $C_1(2i)$ to get,

(A5.6) $$C_1(2i) = (R(2i)+1)^{-1}(d_G(g)G_1(i) + E(g)).$$

Finally, substitute this equation into the expanded expression for $R_g(i)$ and simplify to get,

(A5.7a) $$R^g(i) = \frac{K - G_1(i)}{\frac{E_G}{d_G} + G_1(i)}\left(\frac{1 + R(2i)}{R(2i)}\right).$$

This is the desired equation for $R(i)$ in terms of the future $R(2i)$ and shares $G_1(i)$ held into the state by the *middle-aged*. The transitions to bad states have the identical form:

(A5.7b) $$R^b(i) = \frac{K - B_1(i)}{\frac{E_B}{d_B} + B_1(i)}\left(\frac{1 + R(2i+1)}{R(2i+1)}\right).$$

In functional form, ignoring parameters, these equations are,

(A5.8a) $$R^g(i) = f^g(R(2i), G_1(i)),$$

(A5.8b) $$R^b(i) = f^b(R(2i+1), B_1(i)),$$ with the equality,

(A5.8c) $$R(i) = R^g(i) = R^b(i), \ \forall\ i.$$

Next we derive the recursive relation for shareholding. Again we develop the relation only for good states; the expression for bad states has the same form. Start with the *young*'s budget constraint in a particular successor state (2*i*),

(A5.9) $$C_1(2i) = E(g) - P_G(2i)G_1(2i) - P_B(2i)B_1(2i).$$

Substitute in prices to get,

(A5.10) $$C_1(2i) = E(g) - q_{2i,4i}d_G(g)G_1(2i) - q_{2i,4i+1}d_B(b)B_1(2i).$$

Next substitute the relations for state prices to get,

(A5.11)
$$C_1(2i) = E(g) - dp_{gg}\left(\frac{C_3(4i)}{C_2(2i)}\right)^{-g} d_G(g)G_1(2i) - dp_{gb}\left(\frac{C_3(4i+1)}{C_2(2i)}\right)^{-g} d_B(b)B_1(2i).$$

Use the definition of $R(2i)$ to substitute out $C_2(2i)$, then solve for $C_1(2i)$ to get,

(A5.12) $$\frac{E(g) - C_1(2i)}{C_1(2i)^g} = R(2i)^g d\left[p_{gg}(C_3(4i))^{-g} d_G(g)G_1(2i) + p_{gb}(C_3(4i+1))^{-g} d_B(b)B_1(2i)\right]$$

.

Finally substitute in the *olds*' consumption in terms of the *middle-aged* shares brought into state (4*i*) and (4*i*+1),

(A5.13a) $$\frac{E(g) - C_1(2i)}{C_1(2i)^g} = R(2i)^g d\begin{bmatrix} p_{gg}(d_G(g))^{1-g}(K - G_1(2i))^{-g} G_1(2i) + \\ p_{gb}(d_B(b))^{1-g}(K - B_1(2i))^{-g} B_1(2i) \end{bmatrix}, \text{ where,}$$

$$C_1(2i) = \frac{d_G(g)G_1(i) + E(g)}{R(2i)+1},$$

is the restated budget constraint of the *young*. There is an identical set for the bad state 2*i*+1,

(A5.13b) $$\frac{E(b) - C_1(2i+1)}{C_1(2i+1)^g} = R(2i+1)^g d\begin{bmatrix} p_{bg}(d_G(g))^{1-g}(K - G_1(2i+1))^{-g} G_1(2i+1) + \\ p_{bb}(d_B(b))^{1-g}(K - B_1(2i+1))^{-g} B_1(2i+1) \end{bmatrix}$$

where,

$$C_1(2i+1) = \frac{d_B(b)B_1(i) + E(b)}{R(2i+1)+1}.$$

Substitute $C_1(2i)$ and $C_1(2i+1)$ in the respective equations to get a non-linear recursive relation for shares. In implicit functional form ignoring parameters we have,

(A5.14a) $v^g(G_1(i), R(2i), G_1(2i), B_1(2i)) = 0$, and,

(A5.14b) $v^b(B_1(i), R(2i+1), G_1(2i+1), B_1(2i+1)) = 0; \ \forall i.$

APPENDIX 6: PROOF THAT ASSET PRICES ARE NEGATIVELY AUTOCORRELATED.

Lemma 2: State prices and therefore asset prices are negatively autocorrelated.

Reminders: the state price equation is $q_{2i,4i} = dp_{gg}\left(\frac{C_3(4i)}{C_2(2i)}\right)^{-g}$. From the definition of R we have, $R(i) = \frac{C_2(i)}{C_1(i)} = \frac{C_3(2i)}{C_2(2i)} = \frac{C_3(2i+1)}{C_2(2i+1)}$. We also have, $C_3(2i) = G_2(i)d_g$.

From the state price equation define $Q_{2i,4i}$ and recognize that the state price equates the MRSs of the *young* with the *middle-aged*, with their respective future consumptions.

(A6.1)

$$Q_{2i,4i} = \left(\frac{q_{2i,4i}}{dp_{gg}}\right)^{-\frac{1}{g}} = \frac{C_2(4i)}{C_1(2i)} = \frac{C_3(4i)}{C_2(2i)} = \frac{C_2(4i)+C_3(4i)}{C_1(2i)+C_2(2i)} = \left(\frac{\left(\frac{1}{R(2i)}+1\right)d_{4i}G_2(2i)}{E_{2i}+d_{2i}G_1(i)}\right).$$

This is so because, $\frac{a}{b} = \frac{c}{d} = \frac{a+c}{b+d}$.

(i) From the resource constraint, $C_1(2i)+C_2(2i) = E_{2i} + d_{2i}K - C_3(2i)$,

$C_1(2i)+C_2(2i) = E_{2i} + d_{2i}K - d_{2i}G_2(i) \equiv E_{2i} + d_{2i}G_1(i)$.

(ii) $C_3(4i) = d_{4i}G_2(2i)$, and from $R(2i) = \frac{C_3(4i)}{C_2(4i)}$ write $C_2(4i) = \frac{d_{4i}G_2(2i)}{R(2i)}$.

Use equation (16) to substitute for $E_{2i} + d_{2i}G_1(i)$ and collect terms,

(A6.2) $$Q_{2i,4i} = \left(\frac{\left(\frac{1}{R(2i)}+1\right)d_{4i}G_2(2i)}{\frac{d_{2i}G_2(i)}{R(i)}\left(1+\frac{1}{R(2i)}\right)}\right) = \left(\frac{d_{4i}G_2(2i)}{d_{2i}G_2(i)}R(i)\right).$$

Next, solve for $G_2(2i)$ from the above to get, $G_2(2i) = Q_{2i,4i}\frac{d_{2i}G_2(i)}{d_{4i}R(i)}$, and write equation (16) for $4i$: $\frac{d_{4i}G_2(2i)}{R(2i)}\left(1+\frac{1}{R(4i)}\right) = E_{4i} + d_{4i}G_1(2i)$.

Collect terms,

(A6.3) $$d_{4i}G_2(2i)\left[\left(\frac{1}{R(2i)}\right)\left(1+\frac{1}{R(4i)}\right)+1\right]=E_{4i}+d_{4i}K\,,$$

and substitute in $G_2(2i)$ from above to get,

(A6.4) $$Q_{2i,4i}\frac{d_{2i}G_2(i)}{R(i)}\left[\left(\frac{1}{R(2i)}\right)\left(1+\frac{1}{R(4i)}\right)+1\right]=E_{4i}+d_{4i}K\,.$$

Solve for state prices to get:

(A6.5) $$q_{2i,4i}=p_{gg}d\left(\frac{\frac{d_{2i}G_2(i)}{R(i)}\left[\left(\frac{1}{R(2i)}\right)\left(1+\frac{1}{R(4i)}\right)+1\right]}{E_{4i}+d_{4i}K}\right)^g.$$

Use equation (16) once more to substitute out $R(2i)$ and collect terms to get,

$$q_{2i,4i}=p_{gg}d\left(\frac{\left[\left([E_{2i}+d_{2i}k_1(i)]-\frac{d_{2i}G_2(i)}{R(i)}\right)\left(1+\frac{1}{R(4i)}\right)+\frac{d_{2i}G_2(i)}{R(i)}\right]}{E_{4i}+d_{4i}K}\right)^g.$$

$q_{2i,4i}$ and $R(4i)$ are negatively related; all the other variables are predetermined at state $2i$. When $q_{2i,4i}$ is high $R(4i)$ is low, and since $R(t)$ and $R(t+1)$ are negatively autocorrelated, $R(2i)$ is high; there is a positive relation between $q_{2i,4i}$ and $R(2i)$. QED

TABLE 1
BENCHMARK MODEL PROPERTIES

2-OLG COMPETITIVE, REPRESENTATIVE-CONSUMER, AND 3-OLG SOCIAL PLANNER SOLUTIONS

	2 OLG Competitive		Representative-Consumer		3 OLG SP	
State	**g**	**b**	**g**	**b**	**g**	**b**
ER_m	11.00%	17.04%	-3.51%	4.06%	-4.48%	3.02%
Std(ER_m)	3.02%	3.02%	3.79%	3.79%	3.75%	3.75%
Coef V	0.22	0.22	13.96	13.96	5.12	5.12
Autocor(Er_m)	0.00	0.00	0.00	0.00	0.00	0.00
ER_p	3.15%	6.43%	0.23%	0.97%	0.19%	0.80%
Std(ER_p)	1.64%	1.64%	0.23%	0.97%	0.31%	0.31%
Coef V	0.34	0.34	0.62	0.62	0.62	0.62
Autocor(ER_p)	0.00	0.00	0.00	0.00	0.00	0.00
P_G	0.13	0.04	2.61	0.58	3.19	0.70
Coef V	0.48	0.48	0.64	0.64	0.64	0.64
P_B	0.60	0.19	8.73	1.93	10.67	2.36
Coef V	0.48	0.48	0.64	0.64	0.64	0.64
Coef V (*Div*)	0.15	0.15	0.15	0.15	0.15	0.15
Coef V (*Endow*)	0.15	0.15	0.15	0.15	0.15	0.15

Notes for Tables 1 and 2:

Table 1 displays the properties of the 2-OLG competitive, the representative-consumer, and the 3-OLG Social Planner solutions. Table 2 displays the properties of the 3-OLG competitive solution. The definitions of the variables in the tables are as follows:

- ER_m, is the expected market return.
- ER_p, is the expected risk premium, ER_m - R_f .
- Std(X) is the standard deviation of X.
- In addition to the standard deviations we report coefficients of variation –Coef V-- to make volatilities across some variables and models more comparable. Coef V is computed by dividing the standard deviation by the unconditional mean.
- Autocor(X) is the first order autocorrelation of the X.
- P_s is the price of the risky asset in state s.
- *Div* and *Endow* are dividends, and the endowments of the young, respectively. Since they are identical for the base case of all the solutions, we report their coefficients of variation only in Table 1.

TABLE 2

3 OLG MODEL COMPETITIVE MARKET SOLUTION

Expected Market Return --ER_m					
	gg	**gb**	**bg**	**bb**	**Average**
ER_m	1.78%	9.20%	-0.87%	6.66%	4.16%
Std(ER_m)	12.92%	13.47%	11.02%	11.36%	12.87%
Coef V	7.24	1.46	-12.70	1.71	3.10
Expected Market Risk Premium –ER_p					
	gg	**gb**	**bg**	**bb**	**Average**
ER_p	2.59%	5.45%	1.55%	3.80%	3.34%
Std(ER_p)	2.57%	5.25%	1.62%	3.70%	3.83%
Coef V	0.99	0.96	1.05	0.97	1.15
Price of the *G*-Asset –P_G					
	gg	**gb**	**bg**	**bb**	**Average**
P_G	6.61	1.37	7.33	1.47	4.23
Std(P_G)	7.62	1.57	7.98	1.62	6.31
Coef V	1.15	1.15	1.09	1.11	1.49
Price of the *B*-Asset –P_B					
	gg	**gb**	**bg**	**bb**	**Average**
P_B	25.58	5.78	28.35	6.17	16.61
Std(P_B)	29.07	6.43	30.34	6.62	24.07
Coef V	1.14	1.11	1.07	1.07	1.45

TABLE 3

MODEL AUTOCORRELATIONS

Panel A: Dividend Autocorrelation = 0.00

	ER_m	ER_p	P_G	P_B
1st Order	-0.93	-0.71	-0.44	-0.46

Panel B: Positive Dividend Autocorrelation = +0.60

	ER_m	ER_p	P_G	P_B
1st Order	-0.99	-0.95	-0.37	-0.94

Panel C: Negative Dividend Autocorrelation = -0.60

	ER_m	ER_p	P_G	P_B
1st Order	-0.92	-0.49	-0.45	-0.32

TABLE 4

CORRELATION OF WEALTH WITH CONSUMPTIONS

	Aggregate Consumption	C_1	C_2	C_3
Dividend Autocorrelation 0.00	0.44	-0.76	0.87	-0.64
Dividend Autocorrelation +0.60	0.47	-0.71	0.88	-0.66
Dividend Autocorrelation -0.60	0.50	-0.84	0.88	-0.70

FIGURE 1

The Structure of Uncertainty

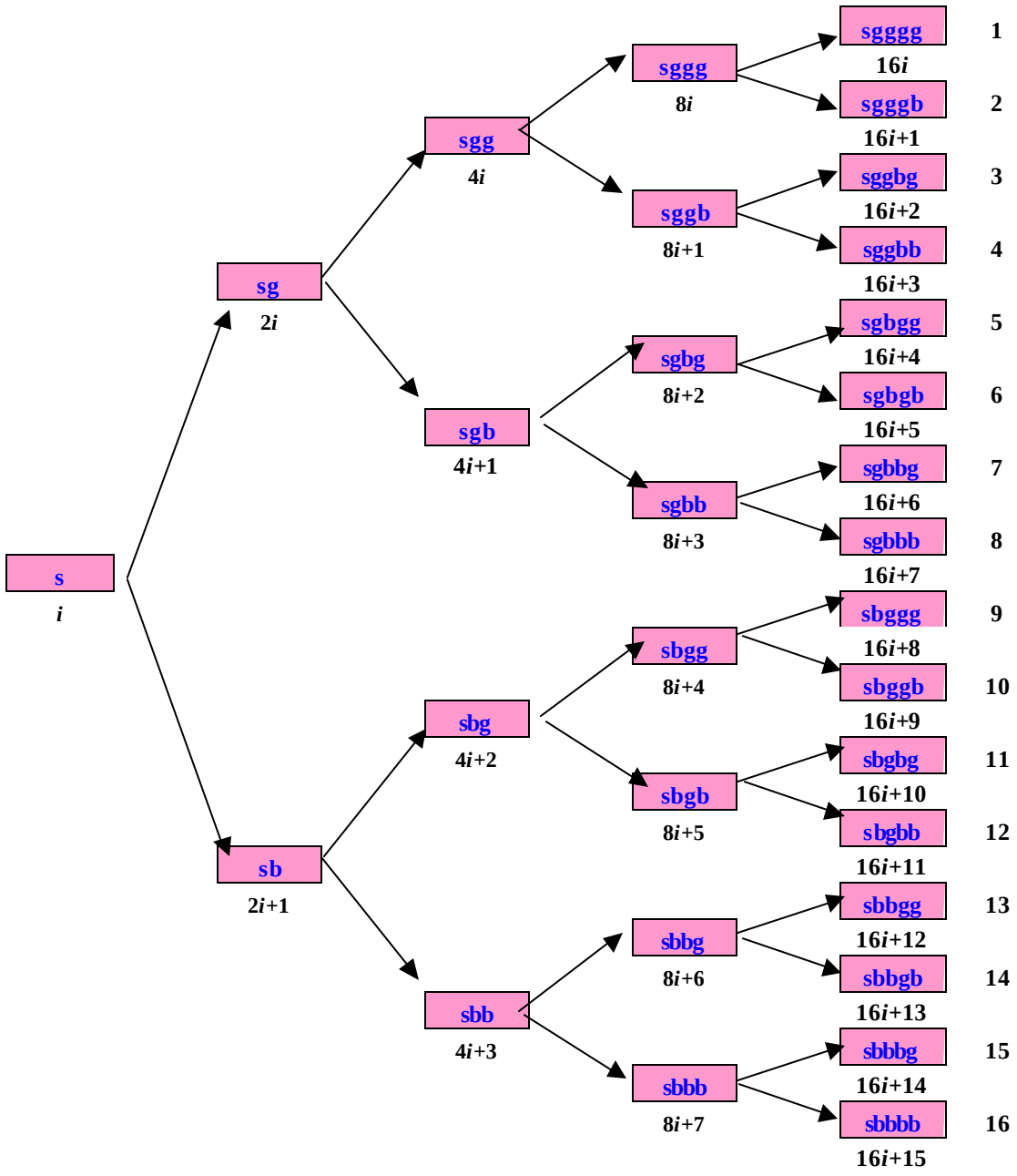

FIGURE 2A: PHASE DIAGRAM OF *R*

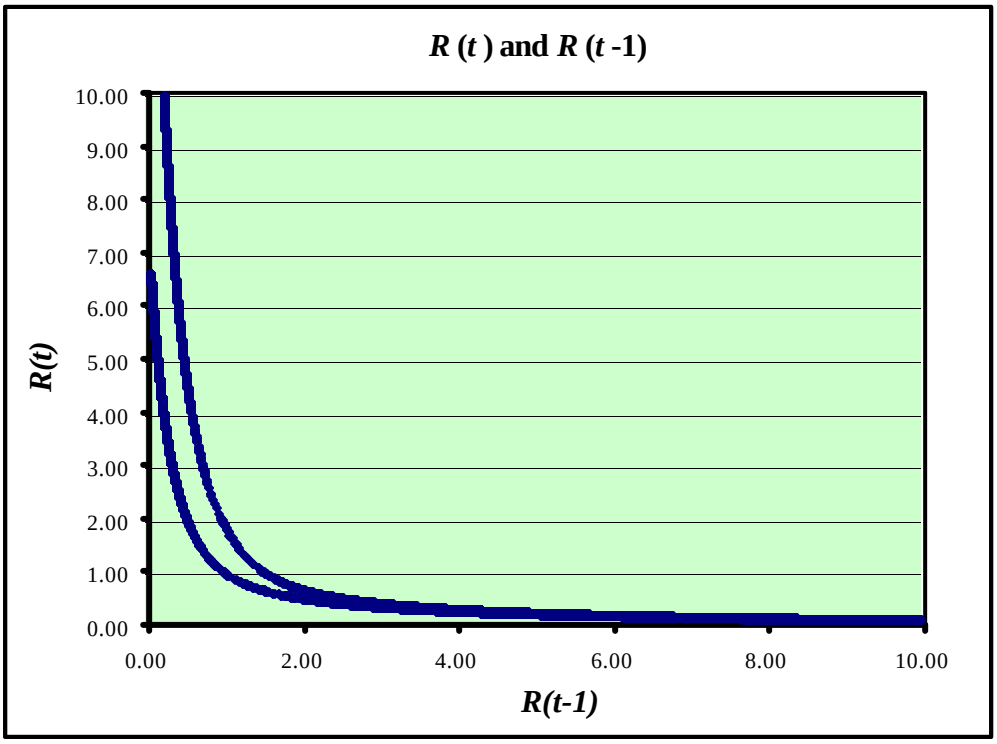

FIGURE 2B: PHASE DIAGRAM OF *ERm*

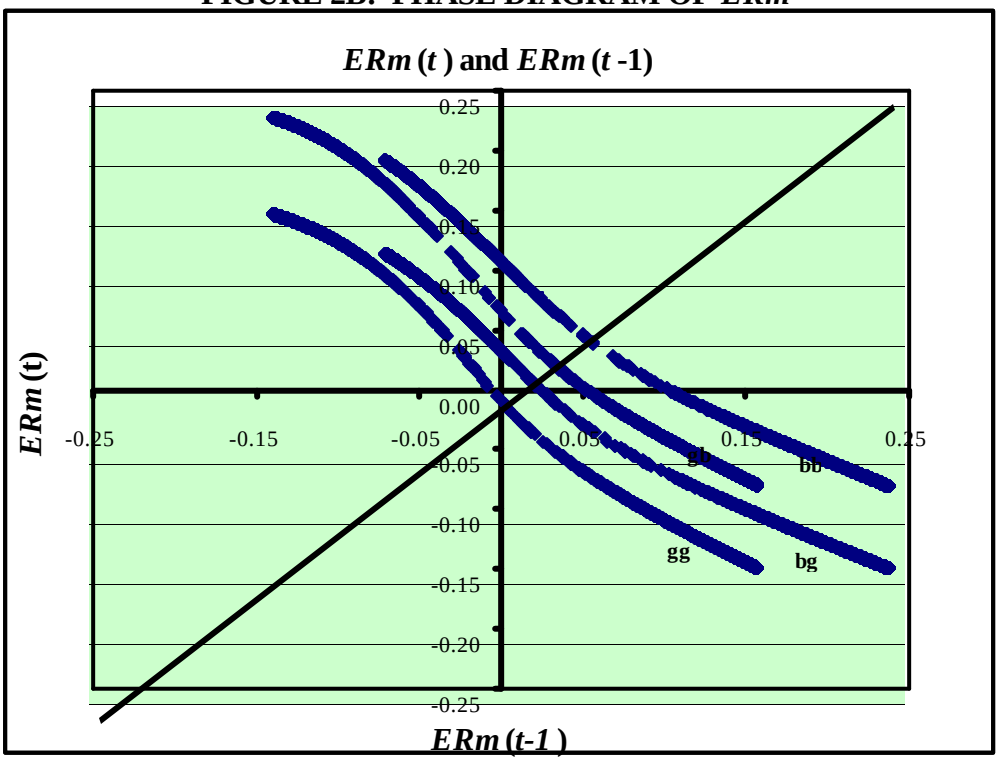

FIGURE 2C: PHASE DIAGRAM OF *ERp*

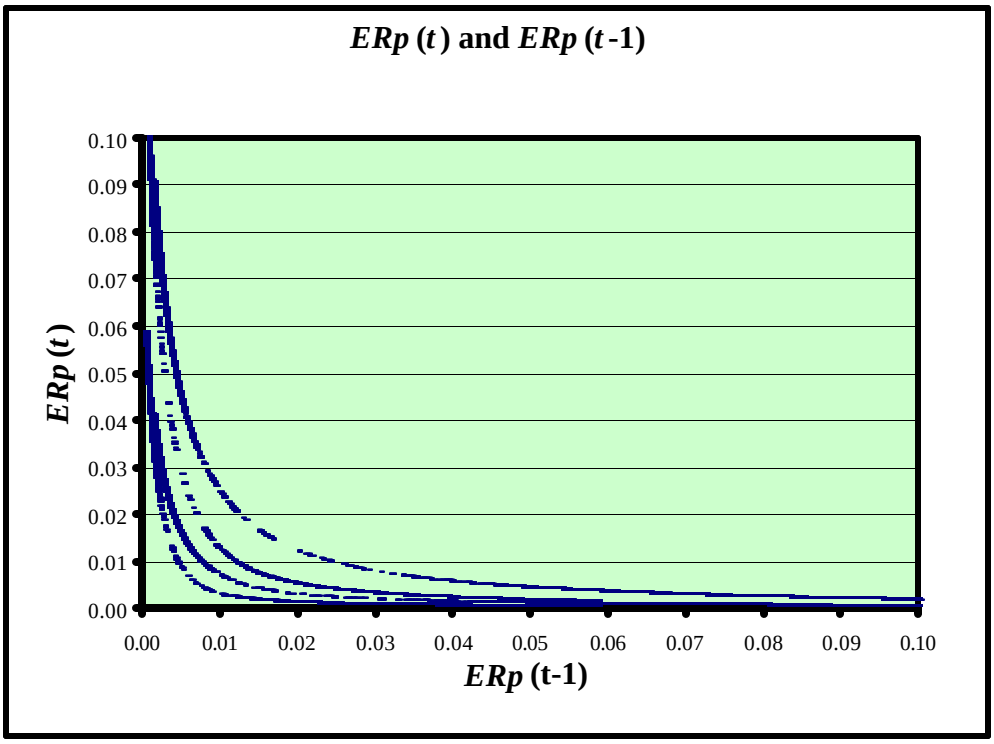

FIGURE 2D: PHASE DIAGRAM OF P_b

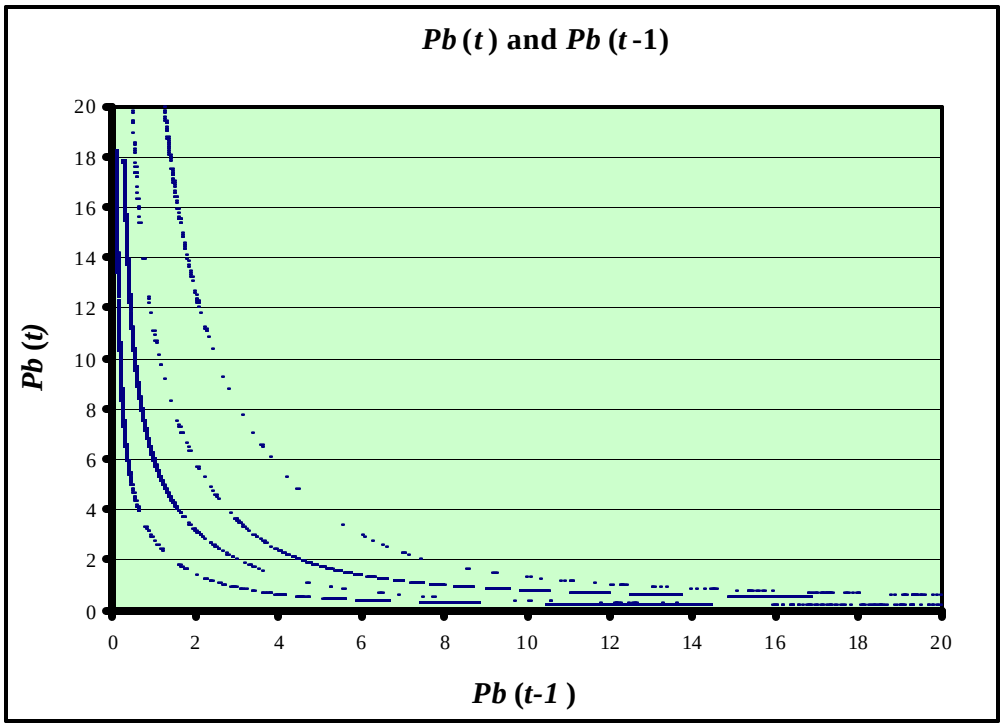

FIGURE 3A

ASSET PRICES AND THE CONSUMPTION OF THE *YOUNG*

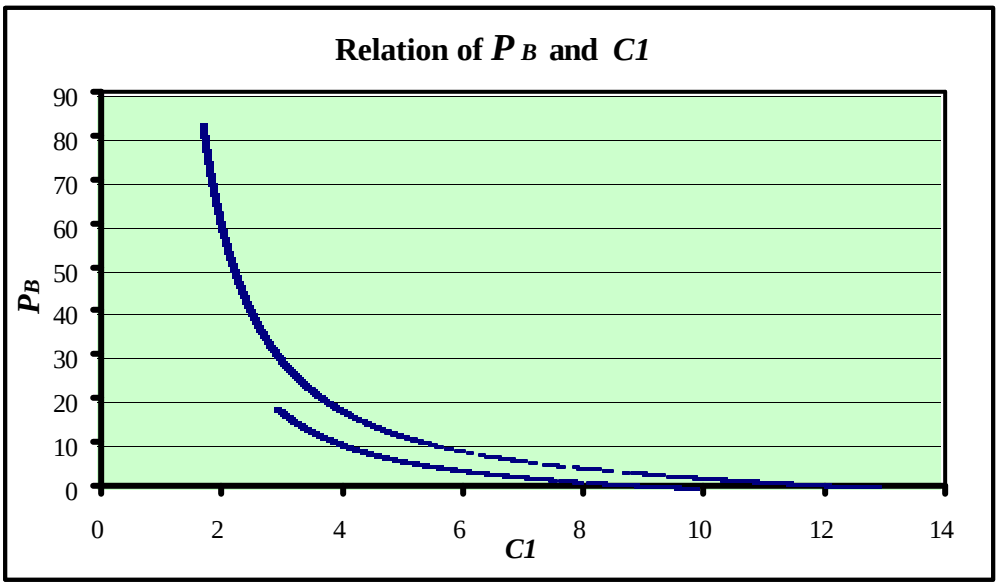

FIGURE 3B

ASSET PRICES AND THE CONSUMPTION OF THE *MIDDLE-AGED*